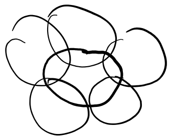
第三讲

2020年3月3日 9:30

$\boxed{(V_\beta, y_\beta)_{\beta\in B}}$ $\{\psi_\beta\}_{\beta\in B}$

- $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$, $a_{ij}\geq 0$



locally finite $\Rightarrow$ at most countably infinite.

M manifold (Hausdorff, second countable)

σ-compact. M = a countable union of compact sets.

Remark 1. $\mathrm{Vol}(K)$ can be defined for any $K\subset M$.

$$\mathrm{Vol}(K) := \sum_\alpha \int_{x^\alpha(K\cap U_\alpha)} \varphi_\alpha\circ x^{\alpha^{-1}} \sqrt{\det(g_{ij}^{x_\alpha})\circ x_\alpha^{-1}}\, dx_\alpha^1\cdots dx_\alpha^n$$

Rie. metric $\Rightarrow$ volume of any subset.

Measure countable additivity

positive linear functional over $C_0^0(M)$

$C_0^0(M) := \{f: M\to\mathbb{R}$, continuous with compact support$\}$

$\mathrm{supp}(f) = \overline{\{x\in M \mid f(x)\neq 0\}}$

vector space $\{U_\alpha, x_\alpha\}_{\alpha\in A}$, $\{\varphi_\alpha\}_{\alpha\in A}$

Define: $$\Lambda f = \sum_\alpha \int_{x_\alpha(U_\alpha)} f\circ x_\alpha^{-1}\cdot\varphi_\alpha\circ x_\alpha^{-1}\sqrt{\det(g_{ij}^{x_\alpha})\circ x_\alpha^{-1}}\, dx_\alpha^1\cdots dx_\alpha^n$$

$x_\alpha(U_\alpha\cap\mathrm{supp}(f))$

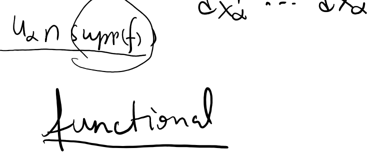
$\forall f\in C_0^0(M)$,

$\Lambda: C_0^0(M)\to\mathbb{R}$ functional

- linear $\Lambda(f+g) = \Lambda f + \Lambda g$ $\Lambda(\lambda f) = \lambda\Lambda(f)$
- positive. $\forall f\in C_0^0(M)$, $f\geq 0 \Rightarrow \Lambda f\geq 0$

Riesz Representation theorem:

Construction of the μ:



V open in M.

V open in M.

$$\mu(V) = \sup\left\{ \Lambda f : \begin{array}{l} f \in C_0^0(M),\ 0 \le f \le 1 \\ \mathrm{supp}(f) \subset V \end{array} \right\}$$

Such a fct always exists in a locally compact Hausdorff topological space. (Urysohn's lemma)

- If $V_1 \subset V_2$ open sets $\Rightarrow \mu(V_1) \le \mu(V_2)$
- $\forall\ E \subset M$, define

$$\mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}$$

If E open, $\mu(E)$ consistent

$$\text{s.t. } \Lambda f = \int_M f\, d\mu \ , \forall f \in C_0^0(M)$$

(M, g)

Thm (Riesz). X topological space, locally compact, Hausdorff, σ-compact.

Let Λ be a positive linear functional on $C_0^0(M)$

Then $\exists$ a σ-algebra $\mathfrak{M}$ which contains all Borel sets, $\exists!$ measure μ on $\mathfrak{M}$ which satisfying

(1) $\Lambda f = \int_X f\, d\mu$, $\forall f \in C_0^0(M)$.

(2) $\mu(K) < \infty$, $\forall$ compact $K \subset X$

(3) μ regular Borel measure.

outer regular, $\mu(E) = \inf\{\mu(V) : V \text{ open}, E \subset V\}$
$\forall E \in \mathfrak{M}$

inner regular $\mu(E) = \sup\{\mu(K), K \subset E \text{ compact}\}$
$\forall E \in \mathfrak{M}$.

Rudin Real and complex analysis.

Rie. metric $\Rightarrow$ volume of subsets $\Rightarrow$ regular Borel measure.

Remark 1. $\mathbb{R}^n$, Lesbegue measure

$$\forall f \in C_0^0(\mathbb{R}^n), \quad \Lambda f = \int_{\mathbb{R}^n} f \, dx \hookrightarrow \text{Riemann integral}$$

Remark 2. (U, x) chart

n-form: $\Omega_0 := \sqrt{\det(g_{ij})}\ \underline{dx^1 \wedge \cdots \wedge dx^n}$

$x \to y \quad \underline{dy^1 \wedge \cdots \wedge dy^n} \longleftrightarrow$ Jacobian $\underline{\det(J(x(p)))}$

atlas $\boxed{\text{orientable}}$. globally defined volume n-form Ω

$$\int_M f \, d\mu = \int_M f \Omega$$

Remark: Complete Rie. manifold M

$$M = \underbrace{\exp_x \Sigma_x}_{\text{chart}} \cup \mathcal{C}(x) \hookrightarrow \text{cut locus}$$

μ - zero measure

$$\int_M f \, d\mu = \int_{\exp_x \Sigma_x} f \, d\mu$$

与何无关

$\hookrightarrow$ function spaces on M.

$L^p(M) :=$ completion of $C_0^0(M)$ w.r.t. $\|\cdot\|_p$.

$$f \in C_0^0(M) \qquad \|f\|_p := \left(\int_M |f|^p \, d\mu\right)^{\frac{1}{p}}$$

$p=2$ $\quad L^2(M)$ Hilbert space

$$\langle f_1, f_2 \rangle_{L^2} := \int_M f_1 f_2 \, d\mu, \quad \forall f_1, f_2 \in L^2(M)$$

$(M, g) \longleftrightarrow \underline{\text{function space}}$

Geometry $\qquad$ Analysis

$\updownarrow$

Topology

休息十分钟，10:50 开始

(II) Geodesics

<u>Introduction</u>. looking for the shortest curves.

$\gamma : [a,b] \longrightarrow M$ regular $|\dot\gamma(t)| \neq 0,\ \forall t \in [a,b]$

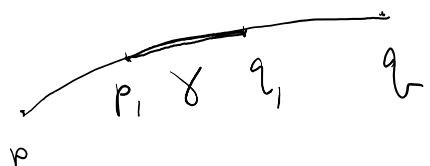
$\cap \mathbb{R}$

$f : M \longrightarrow \mathbb{R}$ function

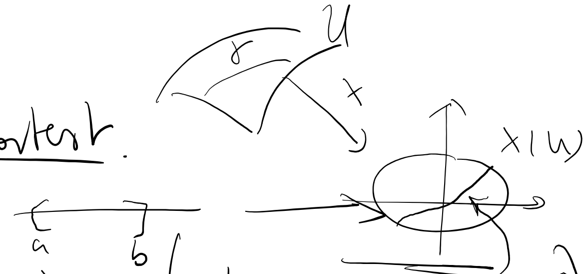
geodesic
harmonic fct $\Big\} \Rightarrow$ harmonic maps.

$$L(\gamma) = \int_a^b \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle}\, dt \qquad \gamma(t) \subset M$$

A chart $\underline{(U, x)}$ $x : U \longrightarrow x(U) \subset \mathbb{R}^n$



<u>shortest</u>.



coordinates of $\gamma(t) \subset M$ $\underline{(x^1(\gamma(t)), \ldots, x^n(\gamma(t)))}$

$$\boxed{\dot\gamma(t) = \dot x^i(\gamma(t)) \frac{\partial}{\partial x^i}} \qquad f : x(U) \longrightarrow \mathbb{R}$$

<u>Proof</u>: $d\gamma\left(\frac{d}{dt}\right) f = \frac{d}{dt}\left(f(\gamma(t))\right) = \frac{d}{dt}\left(f\left(\underline{x^1(\gamma(t)), \ldots, x^n(\gamma(t))}\right)\right)$

$= \sum_i \frac{\partial f}{\partial x^i} \dot x^i(\gamma(t))$, $\forall f \Rightarrow \dot\gamma(t) = \dot x^i(\gamma(t)) \frac{\partial}{\partial x^i}$. $\square$

$\gamma|_{[a,b]} \subset (U, x)$ $\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = g_{ij}(x(\gamma(t)))$

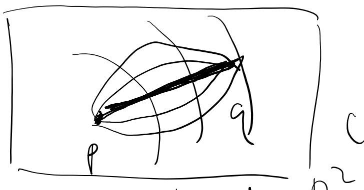
$$\boxed{L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\, \dot x^i(\gamma(t))\, \dot x^j(\gamma(t))}\, dt}$$

$\triangle$

$\gamma(a) = p, \quad \gamma(b) = q$

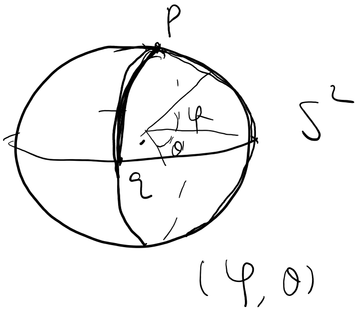
p

$\gamma(a) = p, \quad \gamma(b) = q$

Example: 

(x,y)

straight line $\mathbb{R}^2$



(φ, θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \gamma = \gamma(t) = (r(t), \theta(t)) \quad t \in [a,b]$$

$$\Rightarrow \quad \underline{dr \otimes dr + r^2 d\theta \otimes d\theta}$$

$$\underline{d\varphi \otimes d\varphi + \cos^2\varphi \, d\theta \otimes d\theta}$$

$$\gamma = \gamma(t) = (\varphi(t), \theta(t))$$

$$L(\gamma) = \int_a^b \sqrt{\dot r(t)^2 + \underbrace{r(t)^2 \dot\theta(t)^2}} \, dt$$

$$\dot\theta \equiv 0 \Leftrightarrow = \quad \geq \int_a^b \sqrt{\dot r(t)^2} \, dt = \int_a^b |\dot r(t)| \, dt$$

$$\dot r(t) \Leftrightarrow = \quad \geq \left| \int_a^b \dot r(t) \, dt \right| = \underline{|r(b) - r(a)|}$$

monotone.

$$L(\gamma) = \int_a^b \sqrt{\dot\varphi(t)^2 + \cos^2\varphi(t) \, \dot\theta(t)^2} \, dt$$

$$= \geq \int_a^b \sqrt{\dot\varphi(t)^2} \, dt$$

$$\Updownarrow \quad \dot\theta(t) \equiv 0 = \int_a^b |\dot\varphi(t)| \, dt$$

$$= \geq \left| \int_a^b \dot\varphi(t) \, dt \right| = |\varphi(b) - \varphi(a)|$$

$\Updownarrow$ $\varphi(t)$ monotone.

L. Length functional minimal point

Problem: $\forall \, p, q \in M$.

(i) $\exists$ a shortest curve connecting p and q?

(ii) uniqueness?

§1. Geodesic equation and Christoffel symbols.

Consider. Energy functional. $\gamma : [a,b] \to M$

$$E(\gamma) := \frac{1}{2} \int_a^b \langle \dot\gamma(t), \dot\gamma(t) \rangle \, dt$$

local chart (U, x)

$$= \int_a^b g_{ij}(x(\gamma(t))) \, \dot x^i(\gamma(t)) \, \dot x^j(\gamma(t)) \, dt$$

$$L(\gamma) = \int_a^b \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle} \, dt$$

Remark: $\forall\ \gamma : [a,b] \to M$ C^∞, we have

$$L(\gamma)^2 \leq 2(b-a)E(\gamma)$$

Pf: $L(\gamma)^2 = \left(\int_a^b 1 \cdot \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle}\, dt\right)^2$

$$\overset{\text{Hölder}}{\leq} \left[\left(\int_a^b 1^2 dt\right)^{1/2} \left(\int_a^b \langle \dot\gamma(t), \dot\gamma(t)\rangle\, dt\right)^{1/2}\right]^2$$

$$= (b-a)\, 2E(\gamma). \qquad \square$$

"$=$" $\Leftrightarrow$ $\|\dot\gamma(t)\| \equiv \text{const}$.

$$L(\gamma)^2 = 2(b-a)E(\gamma)$$

arc length. $b-a = L(\gamma)$. $\boxed{L(\gamma) = 2E(\gamma)}$

$L : \gamma \longmapsto L(\gamma) \in \mathbb{R}$

$L : \boxed{(\gamma, \text{ arc length para.})} \longmapsto L(\gamma) \in \mathbb{R}$

$E : (\gamma, \gamma = \gamma(t)) \longmapsto E(\gamma)$

$L = 2E \Big|$ $\{(\gamma, \text{ arc length para})\}$

$\Updownarrow$

parametrized proportionally to arc length

Strategy: find critical point of E over all parametrized curve from p to q.

Lemma: The Euler-Lagrange eq for the energy functional E are

$x(\gamma(t)) = x(t)$

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0, \quad i = 1, \dots, n$$

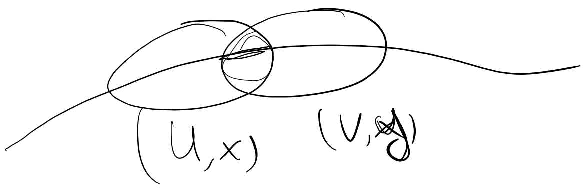
where $\Gamma^i_{jk}(x(t)) = \frac{1}{2} g^{il} \left(g_{lk,j} + g_{kl,j} - g_{jk,l} \right)$

$$g_{lk,j} := \frac{\partial g_{lk}}{\partial x^j}$$

Def. (geodesic) A smooth curve $\gamma : [a,b] \to M$ which satisfies in any (U,x), $x^i(t) = x^i(\gamma(t))$

$$\boxed{\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\,\dot{x}^j(t)\,\dot{x}^k(t)} = 0, \quad i = 1, 2, \dots, n.$$

is called a geodesic.



$y^k(t) := y^k(\gamma(t))$

Exercise: $\Gamma^i_{jk}(x(t)) \sim \Gamma^l_{pq}(y(t))$

tensor?

History: Riemann 1854 1866/68

Christoffel: 1869

Crelles Journal (Journal für die reine und angewandte Mathematik)

Christoffel symbols

Ricci 利玛窦

Italian mathematician Ricci 1883 – 1888

"tensor analysis" "absolute differential calculus"

1901. Ricci, Levi-Civita in French

Klein's journal: Mathematische Annalen

Einstein 1914 derived geodesic equation

Levi-Civita 1916/17 Christoffel symbols

~~ent~~ "parallel transport"

Thm. Let $f \in C^2([a,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$,

Let $x \in C^2([a,b], \mathbb{R}^n)$ be a minimizer

of $I(x) = \int_a^b f(t, x(t), \dot{x}(t))\, dt$

among all functions with $x(a)$, $x(b)$ fixed.

Then x is a solution of the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right) - \frac{\partial f}{\partial x^i} = 0, \quad i = 1, \dots, n.$$

Proof. $\forall\ y \in C_0^2([a,b], \mathbb{R}^n)$

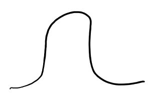
$$I(x + \varepsilon y) = \int_a^b f(t, x(t) + \varepsilon y(t), \dot{x}(t) + \varepsilon \dot{y}(t))\, dt$$

x minimizer $\Rightarrow$ $I(x) \leq I(x + \varepsilon y)$, $\forall \varepsilon$

$$\Rightarrow 0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(x + \varepsilon y) = \int_a^b \frac{d}{d\varepsilon} f(t, x + \varepsilon y, \dot{x} + \varepsilon \dot{y})\, dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x^i} \cdot y^i + \frac{\partial f}{\partial \dot{x}^i} \dot{y}^i\right) dt$$

$$= \int_a^b \frac{\partial f}{\partial x^i} y^i - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right) y^i\, dt$$



$$= \int_a^b \underbrace{\sum_i \left(\frac{\partial f}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right)\right)} y^i\, dt, \quad \forall y^i \in C_0^2([a,b], \mathbb{R}^n)$$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial x^i} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} = 0}_{\text{Euler-Lagrange}}, \quad \forall i = 1, \dots, n. \qquad \square$$

Proof of geodesic eq.: $\gamma \in C^\infty([a,b], \mathbb{R}^n)$

$$E(\gamma) = \frac{1}{2}\int_a^b \sum_{j,k} \underbrace{g_{jk}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t)}\, dt$$

$$E(\gamma) = \frac{1}{2}\int_a^b \sum_{j,k} \underbrace{g_{jk}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t)}_{\parallel \atop f(t,\, x(t),\, \dot{x}(t))}\, dt$$

$\gamma(t) = (x^1(t), \dots, x^n(t))$

$\gamma(a) = p,\ \gamma(b) = q$

E-L eq:

$$\parallel \frac{\partial f}{\partial \dot{x}^i}$$

$$\frac{d}{dt}\left(g_{ik}(x(t))\, \dot{x}^k(t) + g_{ji}(x(t))\, \dot{x}^j(t) \right)$$

$$- g_{jk,i}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0$$

$$\Rightarrow g_{ik,l}(x(t))\, \dot{x}^l(t)\, \dot{x}^k(t) + g_{ik}(x(t))\, \ddot{x}^k(t)$$

$$+ g_{ji,l}(x(t))\, \dot{x}^l(t)\, \dot{x}^j(t) + g_{ji}(x(t))\, \ddot{x}^j(t)$$

$$- g_{jk,i}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0, \quad i = 1, \dots, n$$

$$2 g_{im}(x(t))\, \ddot{x}^m(t) + (g_{ik,j} + g_{ji,k} - g_{jk,i})\, \dot{x}^j \dot{x}^k = 0$$

$i = 1, \dots, n$

Multiply both sides by g^{il}

$$2 \underbrace{g^{il} g_{im}}_{\delta^l_m} \ddot{x}^m + g^{il}(g_{ik,j} + g_{ji,k} - g_{jk,i})\, \dot{x}^j \dot{x}^k = 0$$

$$= 2\ddot{x}^l$$

$$\Rightarrow \ddot{x}^l(t) + \Gamma^l_{jk}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0 \qquad \square$$

Claim. $\gamma(t) = (x^1(t), \dots, x^n(t))$ geodesic

then $\|\dot{\gamma}(t)\| \equiv$ const.

Proof: $\|\dot{\gamma}(t)\|^2 = g_{ij}(x(t))\, \dot{x}^i(t)\, \dot{x}^j(t)$

$$\frac{d}{dt}\left(g_{ij}(x(t))\, \dot{x}^i(t)\, \dot{x}^j(t) \right)$$

$$= g_{ij,k}\, \dot{x}^k(t)\, \dot{x}^i(t)\, \dot{x}^j(t) + g_{ij}(x(t))\, \ddot{x}^i(t)\, \dot{x}^j(t)$$

$$= g_{ij,k}\,\dot{x}^k(t)\,\dot{x}^i(t)\,\dot{x}^j(t) + \underline{g_{ij}(x(t))\,\ddot{x}^i(t)\,\dot{x}^j(t)}$$

$$+ \underline{g_{ij}(x(t))\,\dot{x}^i(t)\,\ddot{x}^j(t)}$$

$$= \boxed{2\,g_{ij}\,\ddot{x}^i}\,\dot{x}^j + g_{ij,k}\,\dot{x}^i\dot{x}^j\dot{x}^k$$

$$= \boxed{2\,g_{lm}\,\ddot{x}^m}\,\dot{x}^l + g_{lj,k}\,\dot{x}^l\dot{x}^j\dot{x}^k$$

$$= -\left(\underline{g_{lk,j} + g_{jl,k} - g_{jk,l}}\right)\dot{x}^l\dot{x}^j\dot{x}^k + g_{lj,k}\,\dot{x}^l\dot{x}^j\dot{x}^k$$

$$= -g_{lk,j}\,\dot{x}^l\dot{x}^j\dot{x}^k + \underline{g_{lj,k}}\,\underline{\dot{x}^l\dot{x}^j\dot{x}^k}$$

$$= 0.$$

$$\Rightarrow \frac{d}{dt}\|\dot{\gamma}(t)\|^2 = 0. \Rightarrow \|\dot{\gamma}(t)\| \equiv \text{const.} \quad \square$$

下课