第四讲

2020年3月5日 13:19

Local Existence and uniqueness of geodesics

$$\gamma : [a,b] \subset \mathbb{R} \longrightarrow U \subset M^n_{g} \xrightarrow{x} x(U) \subset \mathbb{R}^n$$

$$t \longmapsto \gamma(t) \longmapsto x(t) := (x^1(t), \cdots, x^n(t))$$

in (U,x) $\quad t \longmapsto x(t) = (x^1(t), \cdots, x^n(t))$

$\dot{x}(t) = \dot{x}^i(t) \frac{\partial}{\partial x^i}$

If γ is a geodesic, then

$$(\text{Geo}) \quad \ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \quad i = 1, 2, \cdots, n$$

a system of 2^{nd}-order ODEs, nonlinear

ODE: Picard - Lindelöf thm.

$$(\text{Geo}') \begin{cases} \dot{x}^i(t) = y^i(t), & i = 1, 2, \cdots, n \\ \dot{y}^i(t) = -\Gamma^i_{jk}(x(t))y^j(t)y^k(t), & i = 1, 2, \cdots, n \end{cases}$$

More abstractly, denote

$$\vec{X}(t) := (x^1(t), \cdots, x^n(t), y^1(t), \cdots, y^n(t))^T$$

$$(*) \quad \frac{d}{dt}\vec{X}(t) = F(\vec{X}(t)), \quad F \in C^\infty(\underset{x(U)\times\mathbb{R}^n}{\mathbb{R}^{2n}}, \mathbb{R}^{2n})$$

ODE theory: For any $\vec{X}(t_0) = \vec{X}_0$,

there exists $\varepsilon > 0$, s.t. $(*)$ has a unique solution

$$\vec{X}(t), \quad t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

Translate back to (Geo),

For any given $\begin{cases} x^i(t_0) = x^i_0, & i = 1, \cdots, n \\ \dot{x}^i(t_0) = v^i, & i = 1, \cdots, n \end{cases}$

$\exists\ \varepsilon > 0$, s.t. $\overline{(\text{Geo}')}$ has a unique solution

$$x(t) = (x^1(t), \dots, x^n(t)), \quad t \in (t_0 - \varepsilon, t_0 + \varepsilon)$$

Back geometric setting:

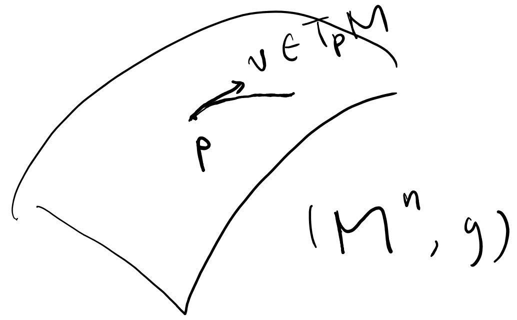
Thm: Let $p \in M$, $v \in T_pM$, then $\exists\ \varepsilon > 0$, s.t.

and a unique geodesic,

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$$

with $\gamma(0) = p$, $\dot{\gamma}(0) = v$.

Space



(M^n, g)

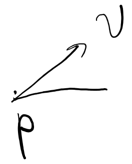
Remark: A geometric interpretation of solutions of (Geo')

$$t \longmapsto (x^1(t), \dots, x^n(t), y^1(t), \dots, y^n(t))$$

can be considered as a curve in TM.

$\forall$ C^∞ curve $\gamma = \gamma(t)$ in M, $\Longrightarrow$

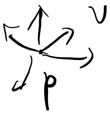
$$(\gamma, \dot{\gamma}) : t \longmapsto (\gamma(t), \dot{\gamma}(t)) \in TM \qquad \square$$



$\boxed{\exists\ \varepsilon} > 0$

Question: given $p \in V \subset M$ (open), given $v \in \underline{B(0, \delta) := \{ v \in T_pM, \|v\| < \delta \}}$

$\exists\ \varepsilon > 0$ s.t.



"uniform"

Thm: [doC, Prop 2.5]

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For $p \in M$, there exists open $V \subset M$, $p \in V$

$(p,0)$ and a $\delta > 0$ (V, δ provides a set of initial data:

$$\mathcal{U}_{V,\delta} := \{ (q, v) : q \in V,\ v \in B(0,\delta) \subset T_qM \})$$

and there exists a $\varepsilon > 0$, and there exists a C^∞ mapping

$(p,0)$

$$\gamma : (-\varepsilon, \varepsilon) \times \mathcal{U}_{V,\delta} \longrightarrow M$$

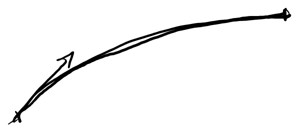
such that for any $(q, v) \in \mathcal{U}_{V,\delta}$, the curve

$$t \longmapsto \gamma(t, q, v), \quad t \in (-\varepsilon, \varepsilon)$$

is the unique geodesic satisfying

$$\gamma(0, q, v) = q, \quad \dot\gamma(0, q, v) = v.$$

Lemma (Homogeneity of a geodesic)



If the geodesic $\gamma(t, q, v)$ is defined on $t\ (-\varepsilon, \varepsilon)$

then the geodesic $\gamma(t, q, \lambda v)$, $\lambda > 0$ is defined on

$t \in \left(-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right)$ and $\quad t \mapsto \gamma(\lambda t, q, v)$

$$\underbrace{\gamma(t, q, \lambda v)}_{\dot\gamma(0,q,\lambda v) = \lambda v}{}_{\in M} = \gamma(\lambda t, q, v), \quad \forall t \in \left(-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right)$$

Proof: $h = h(t) : t \longmapsto \gamma(\lambda t, q, v)$, $t \in \left(-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right)$

Is $h = h(t)$ a geodesic?

$$h(t) := (y^1(t), \dots, y^n(t)) = (x^1(\lambda t), \dots, x^n(\lambda t))$$

γ $\quad t \longmapsto \gamma(t, q, v) \qquad t \in (-\varepsilon, \varepsilon)$

γ: $\boxed{t \longmapsto \gamma(t, q, v)}$ $\qquad t \in (-\varepsilon, \varepsilon)$

Denote it in local coord. $x(t) := (x^1(t), \dots, x^n(t))$

$$\ddot{y}^i(t) + \Gamma^i_{jk}(y(t))\,\dot{y}^j(t)\,\dot{y}^k(t) \overset{?}{=} 0$$

$$= \ddot{x}^i(\lambda t) + \Gamma^i_{jk}(x(\lambda t))\,\dot{x}^j(\lambda t)\,\dot{x}^k(\lambda t)$$

$$= \lambda^2 x''^i(\lambda t) + \lambda^2 \Gamma^i_{jk}(x(\lambda t))\,x'^j(\lambda t)\,x'^k(\lambda t)$$

$$= \lambda^2 (\qquad 0 \qquad)$$

$$= 0$$

$\Rightarrow$ $h = h(t)$ is a geodesic

$$\dot{h}(0) = \lambda \underbrace{x'^i(0)\frac{\partial}{\partial x^i}} = \lambda v$$

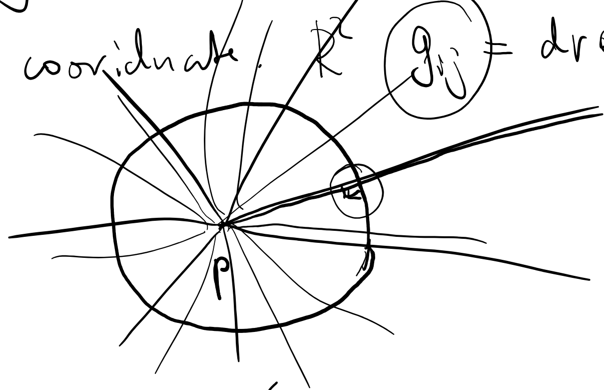
$\Rightarrow$ " = " by the uniqueness of solution $\square$

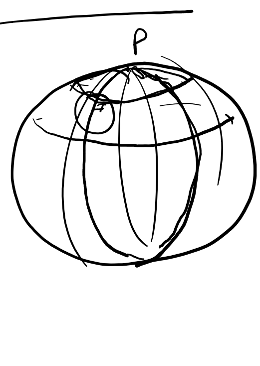
休息到 3:05

§2. Minimizing property of geodesics?

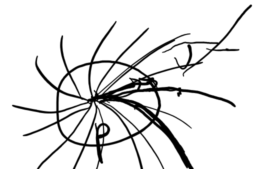
locally shortest curve.

polar coordinate. $\mathbb{R}^2$ $\quad g_{ij} = dr \otimes dr + r^2\, d\theta \otimes d\theta$





(M^n, g)



$v \in T_pM$

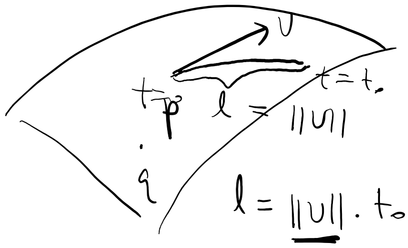
Gauss lemma

Exponential map.

Exponential map:

Def: Let (M^n, g) be a Rie. mfld.

$\forall p \in M, \quad \forall v \in T_pM$



$\exp_p : \cancel{T_pM} \longrightarrow M$

$v \in T_pM \longmapsto \underline{\gamma(1, p, v)} \quad 1 \notin (-\varepsilon, \varepsilon)$

$V_p := \{ v \in T_pM : t \mapsto \gamma(t, p, v) \text{ is defined on } [0,1] \}$

$\subset T_pM \cong \mathbb{R}^n$

is called the <u>exponential map</u> of M at p.

Rmk: ① V_p star shaped around $0 \in T_pM$.

If $v \in V_p$. $\quad \underline{t \mapsto \gamma(t, p, v) \text{ defined on } [0,1]}$

$(-\varepsilon, \varepsilon)$

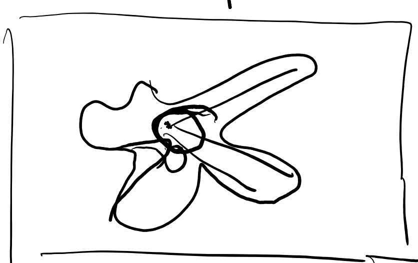
Claim: $\lambda v \in V_p, \quad 0 \le \lambda \le 1$

$t \to \gamma(t; p, \lambda v) \quad \text{on } \left(-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right)$

$p \in V$ $\qquad$ in particular defined on $[0,1]$

② $\forall p \in M, \exists \varepsilon_0 > 0,$

s.t. $B(0, \varepsilon_0) \subset T_pM \subset V_p$



T_pM

Thm. $\forall p \in M, \quad \exists p \in V \overset{\text{open}}{\subset} M, \quad \exists \delta > 0$

$\left(\mathcal{U}_{V,\delta} := \underline{\{ (q, v) : q \in V, \; v \in T_qM, \; \|v\| < \delta \}} \right)$

$\exists \underline{\varepsilon > 0}, \quad \exists \, C^\infty \quad \gamma : \boxed{(-\varepsilon, \varepsilon)} \times \mathcal{U}_{V,\delta} \longrightarrow M$

$\varepsilon > 1$

s.t. $t \mapsto \underline{\gamma(t, q, v)}$ is a geodesic for any $(q, v) \in \mathcal{U}_{V,\delta}$

$\gamma(t, q, (\varepsilon, \ldots)) \qquad (q, v) \in \mathcal{U}_{V,\delta}$

$$t \mapsto \gamma\left(t, q, \frac{\varepsilon}{2}v\right), \quad (q,v) \in \mathcal{U}_{V,\delta}$$

$$t \in \left(-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right) = (-2, 2)$$

$$\lambda = \frac{\varepsilon}{2}$$

$$\left(q, \frac{\varepsilon}{2}v\right) \in \left\{ (q,v) : q \in V, \|v\| < \frac{\varepsilon}{2}\delta \right\}$$

$\Rightarrow$ $\forall p \in M$, $\exists$ $p \in V \overset{\text{open}}{\subset} M$, $\exists$ a $\varepsilon_0 := \frac{\varepsilon}{2}\delta > 0$,

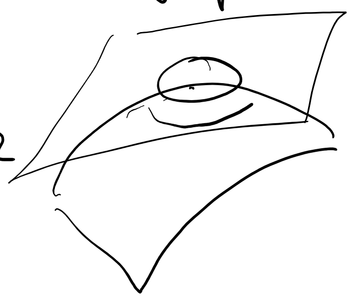
s.t. $\forall q \in V$, then exponential map $\exp_q$

is well defined on $B(0, \varepsilon_0) \subset T_qM$.

<u>Thm2</u>. The $\exp_p$ maps a neighborhood of $0 \in T_pM$ diffeomorphically onto a neighborhood of $p \in M$.

<u>Rmk</u>:

$\exp_p : T_pM \to \mathbb{R}$



<u>Proof</u>: $\exp_p : \Omega \subset \mathbb{R}^n \to \Omega' \subset \mathbb{R}^n$ C^∞

"inverse fct thm" $\exp_p$ is locally diffe. around $0 \in T_pM$

$\Leftrightarrow$ $d(\exp_p)(0)$ non singular.

<u>$d(\exp_p)(0)$</u>

$$\exp_p : T_pM \longrightarrow M$$
$$0 \longmapsto p$$

$$d(\exp_p)(0) : T_0(T_pM) \longrightarrow T_pM$$

$$v \quad \| \quad T_pM$$

$$v \longmapsto d(\exp_p)(0)(v)$$

Pick a curve in T_pM γ s.t $\gamma(0) = 0$, $\dot\gamma(0) = v$ $\quad \gamma(t) = tv$

then $d(\exp_p)(0)(v) = \frac{d}{dt}(\exp_p \circ \gamma(t))\Big|_{t=0}$

$$= \frac{d}{dt}\Big|_{t=0} \exp_p(tv)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma(1, p, tv)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma(t, p, v)$$

$$= v, \quad \forall v \in T_pM$$

$\Rightarrow$ $d(\exp_p)(0) : T_pM \to T_pM$

identity map!

non-singular. $\square$

下课