第六讲

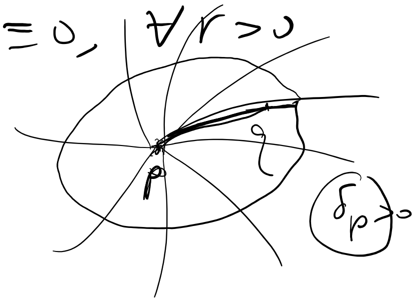
2020年3月12日 13:23

$\forall$ fixed φ

$g_{rr,r}(r,\varphi) = 0, \ \forall r>0 \qquad \lim_{r\to 0} g_{rr}(r,\varphi) = 1$

$g_{r\ell,r}(r,\varphi) = 0, \ \forall r>0 \qquad \lim_{r\to 0} g_{r\ell}(r,\varphi) = 0$

$\Rightarrow g_{rr}(r,\varphi) \equiv 1, \quad g_{r\ell}(r,\varphi) \equiv 0, \ \forall r>0$



$\Rightarrow g_{rr}(r,\varphi) \equiv 1, \ \forall r>0 \ \forall \varphi$

$\qquad g_{r\ell}(r,\varphi) \equiv 0, \ \forall r>0, \forall \varphi$

- Given $p \in M$, $\exists \ \delta_p > 0$, s.t. $\forall q \in M$ satisfying $d(p,q) < \delta_p$

  $\Rightarrow \exists !$ shortest curve from p to q

$\not\Rightarrow$ Given $p \in M$, $\exists \ \delta_p > 0$, s.t. $\forall q_1, q_2 \in B(p, \delta_p)$ metric ball

$\exists !$ shortest curve from q_1 to q_2

Thm (totally normal neighborhood). For any $p \in M$, there exists a neighborhood W of p and a number $\delta > 0$, such that $\forall q \in W$, $\exp_q$ is a diffeomorphism on $B(0,\delta) \subset T_qM$ and $\exp_q(B(0,\delta)) \supset W$.

Pf:

$F: TM \longrightarrow M \times M$

$\quad (q,v) \longmapsto (q, \exp_q v)$

F is local diffeomorphism around $(p,0)$

$\exists \ V, \delta,$ neighborhood

$\mathcal{U} := \{(q,v) \mid q \in V \ \& \ v \in B(0,\delta) \subset T_qM\}$

$\exists$ '' neighborhood

$$\underline{U_{V,\delta} := \{(q,v) \mid q \in V,\ v \in B(0,\delta) \subset T_qM\}}$$

$F|_{U_{V,\delta}}$ is a diffeomorphism $\qquad F((p,0)) = (p,p)$

$W = F(U_{V,\delta})$ is a neighborhood (p,p)

$\exists\ W'$ neighborhood of p in M, s.t. $\underline{W' \times W' \subset W}$

W' is what we want. □

$$\left(\begin{array}{c|c} I & 0 \\ \hline I & I \end{array}\right)$$

- Given a (M^n, g), $\underline{\exists\ \delta_M} > 0$, s.t. $\forall p, q \in M$, if $d(p,q) < \delta_M$, then $\exists!$ shortest curve from p to q. (C^∞, geodesic when assigned parametrization by arc length)

<u>compact</u>

Corollary: Let $\Omega \subset M$ be a compact subset. Then $\exists\ \delta_\Omega > 0$, s.t. $\forall p, q \in \Omega$, with $d(p,q) < \delta_\Omega$ $\exists!$ shortest curve from p to q.

Proof: $\forall p \in \Omega$, $\exists\ \underline{\delta_p} > 0$ s.t. $B(p, \delta_p)$ is a totally normal neighborhood.

$$\Omega \subset \bigcup_{p \in \Omega} B\left(p, \frac{\delta_p}{2}\right) \qquad \Omega \text{ cpt}$$

$$\Rightarrow \exists\ \{p_i\}_{i=1}^{m} \text{ s.t. } \Omega \subset \bigcup_{i=1}^{m} B\left(p_i, \frac{\delta_{p_i}}{2}\right)$$

Set $\delta_\Omega := \min_{i=1,\dots,m} \frac{\delta_{p_i}}{2}$

$\boxed{\forall q \in \Omega,} \Rightarrow \exists\, i$ s.t. $q \in B(p_i, \frac{\delta_{p_i}}{2})$

$B(q, \delta_\Omega)$ is a ~~totally~~ normal neighborhood of q

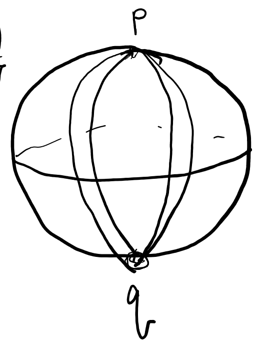
$$B(q,\delta_\Omega) \subset B(q, \frac{\delta_{p_i}}{2}) \subset B(p_i, \delta_{p_i})$$

$\forall q'$, $d(q,q') < \delta_\Omega \Rightarrow \exists!$ shortest curve from q to q'. □

. (M, g) Rie mfld.

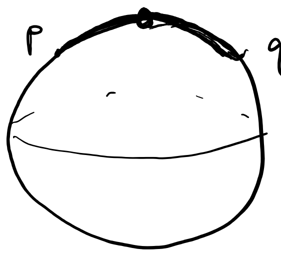
$\forall p, q \in M$, $\exists !$ shortest curve from p to q?

$$d(p,q) = \inf \{ \mathrm{Length}(\gamma) : \gamma \in C_{p,q} \}$$



S^2

Counterexample: $S^2 \setminus \{\text{north pole}\}$



$d(p,q)$

§3. Completeness; Hopf-Rinow Thm.

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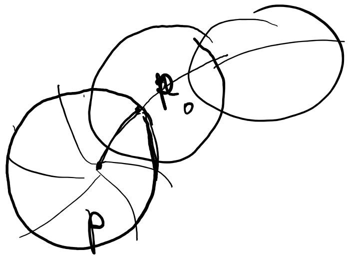
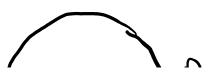


$B_p(\rho) := \exp_p(B(0,\rho) \subset T_pM)$

ρ small normal ball

metric ball



$d(q, \cdot)$ continuous on M, $\partial B_p(\rho) = \underbrace{\exp_p}_{\uparrow\ \text{continuous}}(\underbrace{\partial B(0,\rho)}_{\text{cpt}})$

$\Rightarrow \partial B_p(\rho)$ compact.

$d(q,\cdot)\big|_{\partial B_p(\rho)}$ has minimum value.

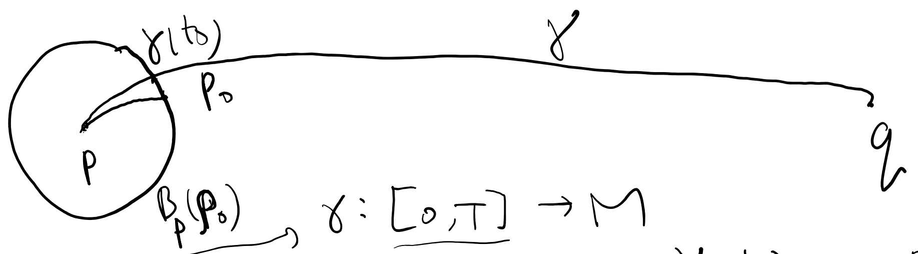
$\exists\ p_0 \in \partial B_p(\rho)$ s.t. $d(q, p_0) = \min\limits_{\tilde{p} \in \partial B_p(\rho)} d(q, \tilde{p})$

$\exists\ V_0 \in T_pM$, $\|V_0\| = 1$, s.t.

$c(t) := \exp_p(tV)$, $t \in [0, \rho_0]$

Claim 1: $d(p, p_0) = \rho_0$, $\boxed{d(p_0, q) = r - \rho_0}$, $r := d(p,q)$

Pf: $d(p_0, q) \geq d(p,q) - d(p, p_0) = r - \rho_0$



$\gamma: [0,T] \to M$

$\gamma \in C_{p,q}$, $\exists\ t_0$ s.t. $\gamma(t_0) \in \partial B_p(\rho_0)$

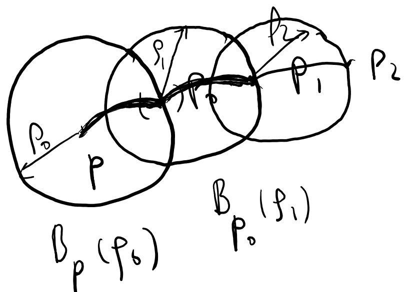
$$\inf_{\gamma \in C_{p,q}} \text{Length}(\gamma) = \text{Length}(\gamma|_{[0,t_0]}) + \text{Length}(\gamma|_{[t_0,T]})$$

$$\geq \rho_0 + \underbrace{d(q, \partial B_p(\rho_0))}_{\geq\, d(q, p_0)}$$

$$= \rho_0 + d(q, p_0)$$

$r = d(p,q) \geq \rho_0 + d(q, p_0)$, $\Rightarrow d(p_0, q) \leq r - \rho_0$ $\square$

$\square$

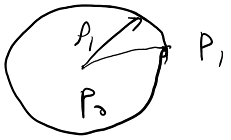


$$d(q,\cdot)\big|_{\partial B_{p_0}(\rho_1)}$$

$$d(q,p_1) = \inf_{\bar{p} \in \partial B_{p_0}(\rho_1)} d(q,\bar{p})$$

Claim 2: $\underline{d(p,p_1) = \rho_0 + \rho_1}$, $\boxed{d(p_1,q) = r - \rho_0 - \rho_1}$

Proof: ~~d(~~ Claim 1 $\Rightarrow$ $d(p_0,q) = r - \rho_0$



$$d(p_0,q) = r - \rho_0$$
$$d(p_1,q) = (r-\rho_0) - \rho_1$$

By the argument of Claim 1,

we know $d(p_1,q) = r - \rho_0 - \rho_1$

$\exists \gamma \in C_{p,p_1}$ s.t. Length$(\gamma) = \rho_0 + \rho_1$ $\Rightarrow$ $d(p,p_1) \leq \rho_0 + \rho_1$

$\triangle$-ineq. $d(p,p_1) \geq d(p,q) - d(p_1,q) = r - (r - \rho_0 - \rho_1)$
$= \rho_0 + \rho_1$

$\Rightarrow d(p,p_1) = \rho_0 + \rho_1$. $\square$

"broken geodesic" is a shortest curve, and hence C^∞.
is a single geodesic

$\boxed{p_1 = \exp_p\left((\rho_0+\rho_1)V_0\right)}$ $\quad t \mapsto \exp_p tV_0$, $t \in [0, \rho_0+\rho_1]$

$p, p_0, p_1, p_2, \ldots\ldots$

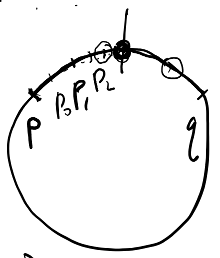
$d(q,p_i) = r - \rho_0 - \rho_1 - \cdots - \rho_i$ $\quad \boxed{\forall \rho_i \geq \delta > 0}$

$\cdots$

$d(q, p_i) = r - \rho_0 - \rho_1 - \cdots - \rho_i$ $\quad \boxed{\forall \rho_i \geq \delta > 0}$

? ρ_i's uniform positive lower bound ?

· (M, g) compact ✓



$S^2 \setminus \{north\}$

<u>Assumption I.</u> $\overline{B_p(r)}$ is compact. $\delta_i \to 0$

p_i's $\quad d(p, p_i) = \sum_{j=0}^{i} \rho_j \leq r$

$\exists\, \delta_{\overline{B_p(r)}}$ s.t. $\delta_i \geq \delta_{\overline{B_p(r)}} > 0, \ \forall i$ ✓

<u>Assumption II</u> (M, g) is <u>complete</u> as a metric space

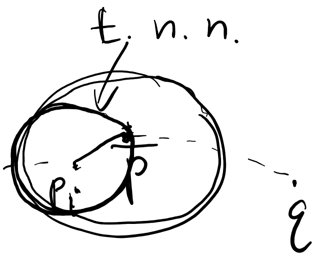
$\delta_i \to 0$

p_i's Cauchy sequence $\forall \varepsilon > 0, \exists N$

if $i, j \geq N$, $d(p_i, p_j) < \varepsilon$

<u>Assume</u> δ_i's $\quad \delta_i \to 0$

completeness $p_i \to \bar{p} \in M$



$\exists\, i_0, \ \exists\, \delta_{i_0}$ s.t.

$\bar{p} \in \partial B_{p_{i_0}}(\delta_{i_0})$

$d(\bar{p}, q) = \lim_{i \to \infty} d(p_i, q) = \underline{r - \sum_{i=0}^{\infty} \delta_i}$

$\Rightarrow \quad d(p, \bar{p}) = \sum_{i=0}^{\infty} \delta_i$

<u>Assumption III.</u> <u>$\exp_p$ is defined on the whole T_pM.</u>

δ_i's $\to 0 \quad \Rightarrow$ <u>p_i's</u> Cauchy sequence.

v_i's ... $\Rightarrow$ $\{p_i\}$ Cauchy sequence.

$\exp_p t v_0$, $\quad v_0 \in T_pM$, $\|v_0\| = 1$

$t \in [0, \sum_{i=0}^{\infty} \delta_i]$

is defined on $t \in [0, \infty)$

$[0, \infty)$ complete

$\sum_{j=0}^{i} \delta_j$ Cauchy sequence in $[0, \infty)$

$\exists \ t_0 = \sum_{j=0}^{\infty} \delta_j \qquad \underbrace{\exp_p t_0 v_0}_{p_i\text{'s}} \longrightarrow q$

$d(p_i, q) \searrow 0$

A more elegant proof.

$c: t \mapsto \exp_p t v_0$ is defined on $[0, \infty)$

Consider. $I := \{ t \in [0, r] : d(\underset{\exp_p t v_0}{\underline{c(t)}}, q) = \overset{d(p,q)}{r} - t \}$

if $r \in I \Rightarrow d(c(r), q) = r - r = 0$ ✓

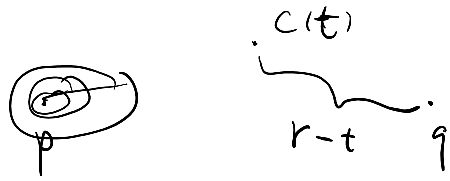
$I = [0, r] \Leftarrow$

① $I \neq \phi \qquad 0 \in I$

② I closed $\qquad f := d(c(t), q) - r + t \in C^0$

$I = f^{-1}(0) \cap [0, r]$ closed.

③ I open.

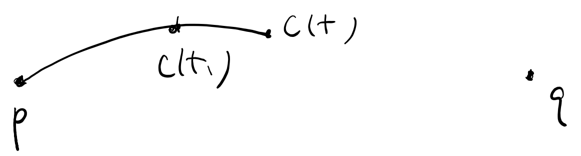


$\forall \ t \in I,$

$\exists \delta$ s.t. $[t, t+\delta) \in I$

$\forall\, t_1 \in [0,t]$, we have $t_1 \in I$.

t_1



$t \in I \Rightarrow d(c(t),q) = r - t$

$$d(c(t_1),q) \leq \underbrace{d(c(t_1),c(t))} + \underbrace{d(c(t),q)}$$

$$\overset{c \text{ shortest}}{=} t - t_1 + r - t$$

$$= r - t_1$$

$$d(c(t_1),q) \geq d(p,q) - d(p,c(t_1))$$

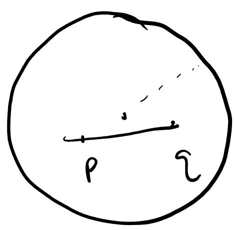
$$= r - t_1$$

$$\Rightarrow d(c(t_1),q) = r - t_1. \Rightarrow t_1 \in I.$$

In conclusion, $\forall\, t \in I$, $\Rightarrow [0, t+\delta) \in I$.

$\Rightarrow I$ open. $\square$

Hopf - Rinow.



$B(0,1) \subset \mathbb{R}^n$

下课.