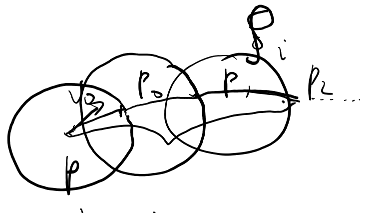
第七讲

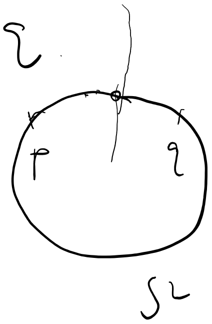
2020年3月17日 8:30

$\forall p, q \in M$



$d(q, p_i) \downarrow$

existence of shortest curve from p to q.



"broken geodesic" C^∞

$$\sum_{i=1}^{\infty} \rho_i < \infty$$

S^2

"complete" $\begin{cases} \cdot\ M \text{ complete metric space} \\ \cdot\ r = d(p,q),\ \overline{B_p(r)} \text{ compact} \\ \cdot\ \exp_p : T_pM \to M \text{ is defined on all } T_pM \end{cases}$

$\exp_p tv_0$ is defined $t \in [0, \infty)$

<u>Thm</u>. (Hopf - Rinow 1931)

Let (M, g) be a Rie. mfld. TFAE:

(i) M is a complete metric space.

(ii) The <u>closed</u> and <u>bounded</u> subsets of M is compact.

(iii) $\exists p \in M$ for which $\exp_p$ is defined on all T_pM.

(iv) $\forall p \in M$ for which $\exp_p$ is defined on all T_pM.

Each of the statements (i) - (iv) implies

(v) $\forall p, q \in M$, $\exists$ a shortest curve from p to q.

(i.e. a C^∞ curve from p to q with length $= d(p,q)$,

When parametrized prop. to arc length, it is a geodesic)

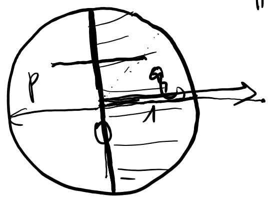
<u>Remark</u>: (1) „All concepts of completeness are equivalent".

Remark: (1) „All concepts of completeness are equivalent".

Hopf, Rinow. 1931. Commentarii Mathematici Helvetici

Comm. Math. Helv. 1929 founded

(2). $\mathbb{R}^n$



$B(0,1) \subset \mathbb{R}^n$ open ball, Euclidean metric

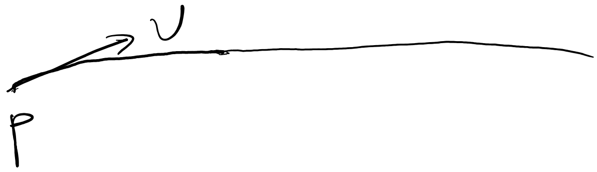
Rie. manifold.

convex.

$$\exp_0 : \underset{\substack{\mathbb{R} \\ T_0\mathbb{R}^n \cong \mathbb{R}^n}}{T_0(B(0,1))} \to B(0,1)$$

(3). "geodesically completeness": A Rie. mfld M is geodesically complete if $\forall p \in M$, $\exp_p$ is defined on all T_pM.

$\forall p \in M$, $\forall v \in T_pM$. $\exp_p tv$ is defined on $t \in [0,\infty)$



(4). Heine-Borel property $\mathbb{R}^n$

discrete set $S := \{ a_i, i=1,2,\dots \}$

$$d : S \times S \to [0,\infty)$$

$$d(a_i, a_j) = \delta_{ij}$$

Complete metric space

bounded closed. not cpt.

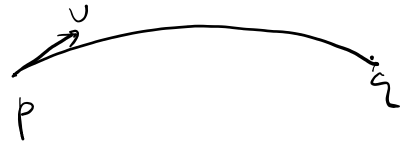
bounded closed. not cpt.

(5). $\boxed{\text{complete}}$ Rie. mfld. (M, g)

$$\exp_p : T_pM \longrightarrow M \text{ is surjective.}$$

$\forall q \in M$, $\exists$ geodesic γ from p to q

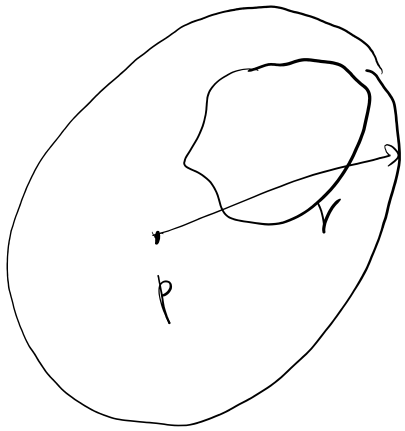
$q \in \exp_p(T_pM)$.



Proof: $(iv) \underset{\text{trivial}}{\Longrightarrow} (iii) \Longrightarrow \underbrace{(ii) \Longrightarrow (i)}_{\text{standard}}$

(iii) $\Rightarrow$ (ii) Let $K \subset M$, closed and bounded.

"bounded" $\Rightarrow$ $\exists p \in M$, $\exists r > 0$ s.t. $K \subset \overline{B_p(r)}$ $\to$ metric ball.



If $\overline{B_p(r)}$ is compact,

then any closed subset of $\overline{B_p(r)}$ is compact.

$$\exp_p : \underset{\substack{\cup\!\!| \\ B(0,r)}}{\overset{\cong \mathbb{R}^n}{T_pM}} \longrightarrow M$$

Claim: $\exp_p\left(\overline{B(0,r)}\right) = \overline{B_p(r)}$ $\quad C^\infty$ map

$$\exp_p\left(\overline{B(0,r)}\right) \subseteq \overline{B_p(r)}$$

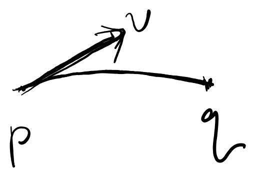
$\forall v \in \overline{B(0,r)}$. $\quad \exp_p v = \gamma(1, p, v)$

$$\text{Length}\left(\gamma|_{[0,1]}\right) = \|v\| \leq r$$

$$\underbrace{\mathrm{Length}\left(\gamma|_{[0,1]}\right)}_{\geq\; d(p,\ \exp_p v)} = \|v\| \le r$$

$\Rightarrow \exp_p v \in \overline{B_p(r)}$

$\forall\, q \in \overline{B_p(r)}$,



$d(p,q) \le r$

$\Rightarrow \exists\, v$, $\underline{\|v\| \le r}$, s.t. $\exp_p v = q$.

$\overline{B_p(r)} \subseteq \exp_p\left(\overline{B(0,r)}\right)$. $\square$

$\Rightarrow \overline{B_p(r)}$ is compact. $\square$

(ii) $\Rightarrow$ (i) standard.

$\forall$ Cauchy sequence $(p_n)_{n\in\mathbb{N}}$.

$\left(\forall \varepsilon>0, \ \exists N, \text{ s.t. whenever } n,m>N, \ \underline{d(p_n,p_m)} < \varepsilon \right)$

<u>Claim</u>: $\forall\, p_0 \in M$. $\left(d(p_n,p_0)\right)_{n\in\mathbb{N}}$ is a Cauchy seq.

$$\left| d(p_n,p_0) - d(p_m,p_0)\right| \le \underline{d(p_n,p_m)}$$

<u>$\mathbb{R}$ complete</u>. $\Rightarrow \lim\limits_{n\to\infty} d(p_n,p_0)$ exists.

$(p_n)_{n\in\mathbb{N}}$.

① If $(p_n)_{n\in\mathbb{N}}$ has an accumulative point, $\exists$ subsequence $(p_{n_k})_{k\in\mathbb{N}}$ s.t. $p_{n_k} \longrightarrow a_0 \in M$ as $k\to\infty$

Set $p_0 = a_0$ in the Claim.

$\lim\limits_{n\to\infty} d(p_n, a_0)$ exists.

$\lim\limits_{k\to\infty} d(p_{n_k}, a_0) = 0$

$\lim_{n\to\infty} d(p_n, a_0)$ exists. $\quad \lim_{k\to\infty} d(p_{n_k}, a_0) = 0$

$\Rightarrow \lim_{n\to\infty} d(p_n, a_0) = 0$. i.e. $p_n \to a_0 \in M$ as $n\to\infty$.

② If $(p_n)_{n\in\mathbb{N}}$ has no accumulative pt.

~~$(p_n)_{n\in\mathbb{N}}$~~ $\{p_n : n\in\mathbb{N}\}$ is closed.

Cauchy sequence $\Rightarrow$ $\{p_n : n\in\mathbb{N}\}$ is bounded.

HB (ii) $\Rightarrow$ $\{p_n : n\in\mathbb{N}\}$ compact. $\lightning$

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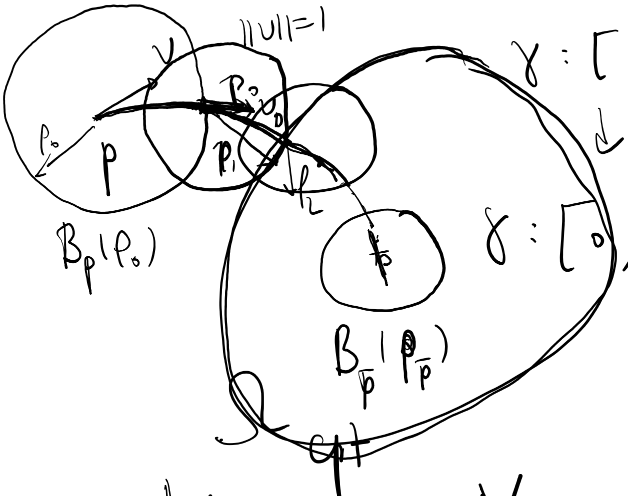
(i) $\Rightarrow$ (iv) $\quad (M, g)$ complete metric space.

Aim: $\forall p$, $\forall v \in T_pM$ with $\|v\|=1$,

$t \mapsto \exp_p(t\underline{v})$ is defined on $t\in[0,\infty)$



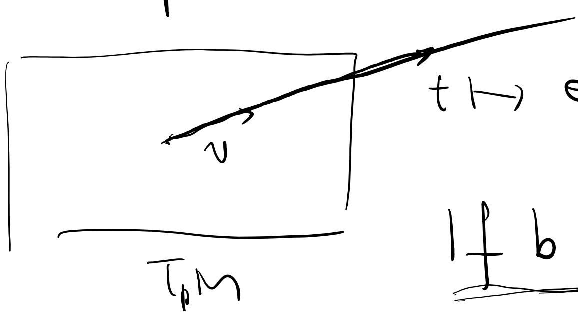
$\gamma : [0, \rho_0] \longrightarrow M$

extension (local existence and uniqueness of geodesics)

$\gamma : [0, \rho_0+\rho_1+\rho_2] \to M$

$\sum_{i=0}^{\infty} \rho_i < \infty$

$t \mapsto \exp_p tv \qquad V$ ~~$\in$~~ $\{tV, t\in[0,1]\}$ star-shaped



$t\mapsto \exp_p tv : \quad t\in[0, b)$ (ODE theory)

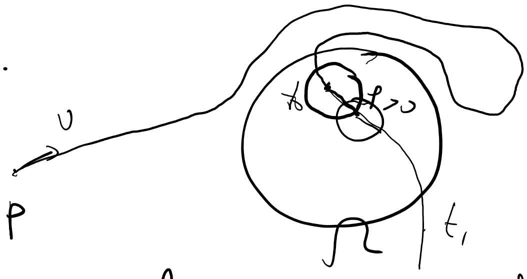
maximal interval. $b = \infty$

If $b<\infty$ $\overset{?}{\Rightarrow}$ contradiction?

T_pM If $b < \infty \Rightarrow$ contradiction?

Claim: Suppose $b<\infty$. $\forall$ compact $\Omega \subset M$, with $\exists t_0 \wedge \exp_p t_0 v \in \Omega$, then $\exists\, t_1 \in (t_0, b)$ s.t.

$$\exp_p t_1 v \notin \Omega.$$



Proof: Ω cpt $\Rightarrow \exists\, \delta_\Omega > 0$ s.t. $\forall\, q \in \Omega$, $\overline{B_q(\delta_\Omega)}$ is a normal neighborhood of q

~~$\forall\, t \in [t_0, b)$~~ If $\forall\, t \in [t_0, b)$, $\exp_p tv \in \Omega$.

Then $\forall\, t \in [t_0, b)$, $\exists\, t + \delta_\Omega \in [t_0, b)$

Recall $b < \infty$ ↯. $\square$

$t \mapsto \exp_p tv \quad t \in [0, b)$, maximal interval, $b < \infty$, $\in \mathbb{R}$

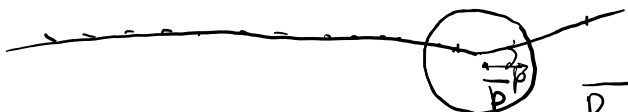
$\exists\ (t_n)_{n\in\mathbb{N}}$ s.t. $t_n \to b$ as $n \to \infty$

$\Rightarrow \left(\exp_p t_n v\right)_{n\in\mathbb{N}}$ $\quad d(\gamma(t_n), \gamma(t_m)) \le |t_n - t_m|$

$\quad = \gamma(t_n)$

$\Rightarrow$ Cauchy seq. $\Rightarrow \exists\ \bar{p} \in M$ s.t.

$$\boxed{\gamma(t_n) \longrightarrow \bar{p} \quad \text{as } n \to \infty}$$



$\overline{B_{\bar p}(\delta_{\bar p})}$ normal ball

$\exp_{\bar p}: \overline{B(0, \delta_{\bar p})} \subset T_{\bar p}M \to \overline{B_{\bar p}(\delta_{\bar p})}$

Claim $\exists N$ s.t. $\forall\, n \ge N$ diffeomorphism

$\text{s.t. } \varphi_{\bar{p}} : B(0, \delta_{\bar{p}}) \subset T_{\bar{p}}M \to B_{\bar{p}}(\delta_{\bar{p}})$ diffeomorphism

Claim ~~∃~~ $\forall N$ ∃ $n > N$ s.t. $d(\gamma(t_n), \bar{p}) \geq \delta_{\bar{p}}$ ↯. □ qed

Complete Rie. manifold.

Compact Rie. manifold. Any closed subset is compact

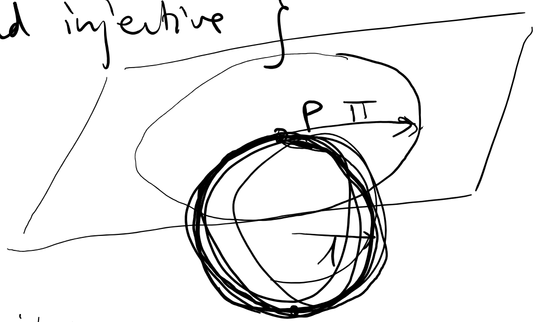
Injective radius: (M, g) Rie. manifold.

The injective radius of $p \in M$ is

$$i(p) := \sup \{ \rho > 0 : \exp_p \text{ is defined } B(0,\rho) \subset T_pM \text{ and injective} \}$$

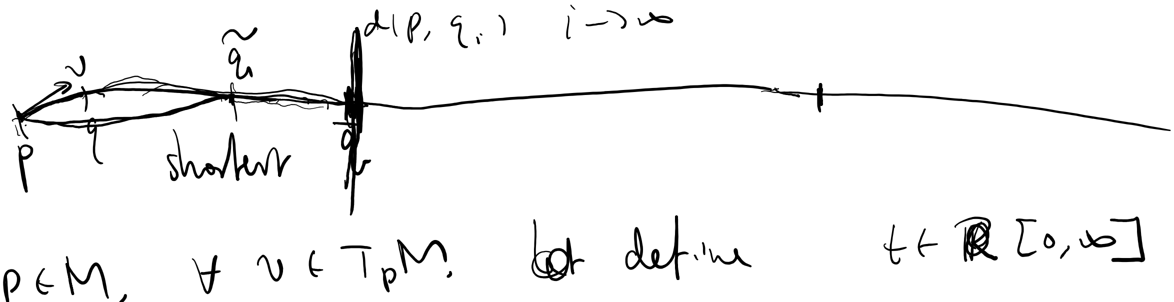
The injective radius of M is

$$i(M) := \inf_{p \in M} i(p)$$



$i(p) = \pi$ S^2

Is a geodesic shortest?



$\forall p \in M$, $\forall v \in T_pM$, define $t \in [0, \infty]$

$$\text{cut}(v) = \max \{ t : \text{The geodesic } \gamma(t,p,v) = \exp_p(tv) \text{ is the shortest from } p = \gamma(0,p,v) \text{ to } \gamma(t,p,v) \}$$

define $C_p := \{ tv : v \in T_pM, 0 \leq t \leq \text{cut}(v) \}$

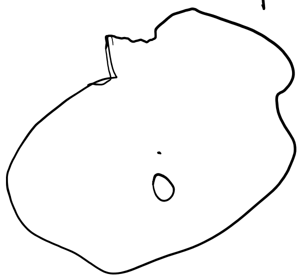
$\wedge$

define. $C_p := \{ tv : v \in T_p\ldots, 0 \le t \le \mathrm{cut}(v) \}$
$\cap$
T_pM star shaped, $0 \in C_p$.

~~ex~~: completeness $\Rightarrow$ $\exp_p : T_pM \longrightarrow M$ surj.

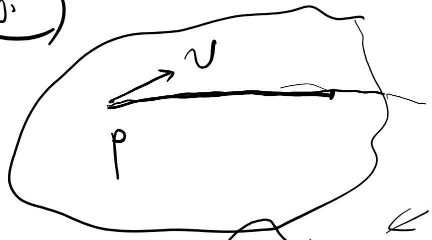
Proposition: $\exp_p$ maps C_p surjectively onto M.
i.e. $\exp_p(C_p) = M$.

Pf: Hopf-Rinow. $\forall q \in M$, $\exists$ shortest geodesic from p to q. $\exists\ v \in C_p$ s.t. $\exp_p v = q$. $\square$



$C_p \subset T_pM \cong \mathbb{R}^n$

Definition: We call $\exp_p(\partial C_p) \subset M$ the cut locus of p.
(割迹)



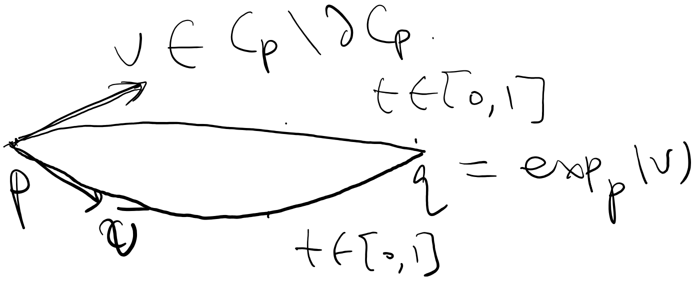
$M = \exp_p(\partial C_p) \sqcup \exp_p(C_p \backslash \partial C_p)$

open

Claim: $\forall q \in \underbrace{\exp_p(C_p \backslash \partial C_p)}_{\text{interior of cut locus}}$,
$\exists !$ shortest geodesic from p to q.

If not unique,

$\Rightarrow$ $\boxed{\exp_p : C_p \setminus \partial C_p \to M \quad \text{injective}}$



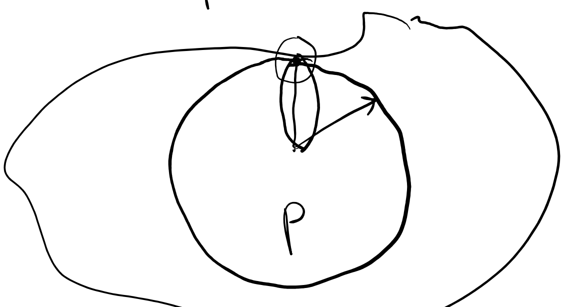
If not injective, $\exists\ \bar{v} \in C_p \setminus \partial C_p$, $\bar{v} \neq v$

s.t. $\exp_p \bar{v} = q \xRightarrow{\text{claim}} \bar{v} = v$ $\lightning$

Injective radius of $p \in M$

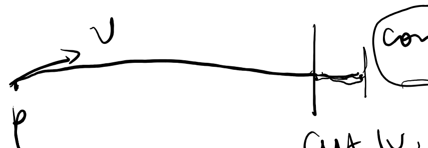
$$i(p) := d(p, \exp_p(C_p))$$

$$i(M) = \inf_{p \in M} i(p) \qquad [\text{Gallot - Hulin - Lafontaine}]$$



$\underline{d\exp_p}$ singular

· $i(p) = \sup\{\rho > 0 : \exp_p \text{ is a diffeomorphism on } B(0,\rho) \subset T_pM\}$



conjugate point

伍鸿熙

· A C^∞ mapping $F : U \xrightarrow{C\mathbb{R}^k} F(U)$ is a C^∞ diff. iff F is <u>injective</u> and <u>dF is nonsingular at any point of U.</u>

- $\Omega \subset M$ cpt, $\exists \delta_\Omega > 0$ s.t. $\forall q \in \Omega$, $B_q(\delta_\Omega)$ is a normal neighborhood.

$\Rightarrow \exists \delta_\Omega > 0$ s.t. $i(\Omega) := \inf_{p \in \Omega} i(p) \geq \delta_\Omega > 0$.

<u>Local isometry</u> : $\pi : \mathbb{R}^n \to T^n$ local isometry.

<u>Def</u>: A diffeomorphism $h: M \to N$ is an <u>isometry</u> if it preserve the Rie. metric. That is, $\forall p \in M$, $v_1, v_2 \in T_pM$, we have $\underline{dh}: T_pM \to T_{h(p)}N$

$$\boxed{g_M(v_1, v_2) = g_N(dh(v_1), dh(v_2))}$$

A C^∞ map $h: M \to N$ is a <u>local isometry</u> if $\forall p \in M$, $\exists$ neighborhood U of p for which

$$h|_U : \underset{\underset{M}{\cap}}{U} \to h(U) \subset N$$

is an isometry, and $h(U)$ is open in N.

- $p \in M$. (U, x) (g_{ij})

$h(p) \in N$, (V, h) $(\gamma_{\alpha\beta})$

$(h^1(x_1, \dots, x_n), h^2(x_1, \dots, x^n), \dots, h^n(x_1, \dots, x_n))$

$$\boxed{g_{ij}(q) = \gamma_{\alpha\beta}(h(q)) \frac{\partial h^\alpha(q)}{\partial x^i} \frac{\partial h^\beta(q)}{\partial x^j}}$$

- A map $F: (M, g_M) \to (N, g_N)$ is a local ~~Rie.~~ ~~manifold~~, isometry,

$\Rightarrow \forall p \in M$, $dF(p) : T_pM \to T_{F(p)}N$

is a linear isometry.

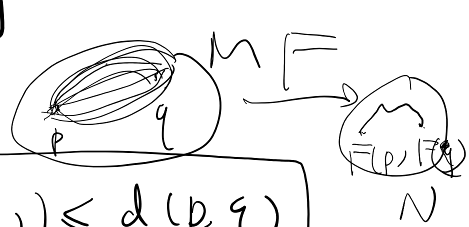
<u>Prop.</u> Let $F : (M, g_M) \to (N, g_N)$ be a <u>local isometry</u>

(1) F maps geodesics to geodesics.

(2). $\underline{F \circ \exp_p(v) = \exp_{F(p)} \circ dF(p)(v)}$ whenever $\exp_p(v)$ is defined.

$$\begin{array}{ccc} \mathcal{O}_p \subset T_pM & \xrightarrow{dF(p)} & \mathcal{O}_{F(p)} \subset T_{F(p)}N \\ \downarrow \exp_p & & \downarrow \exp_{F(p)} \\ M & \xrightarrow{F} & N \end{array}$$

(3). F is distance decreasing:



$\forall\, p, q \in M$, $\boxed{d(F(p), F(q)) \le d(p, q)}$

(4) If F is also bijection, then it is distance preserving, i.e. $\underline{d(F(p), F(q)) = d(p, q)}$

<u>Proof</u>: (1) M. (U, x) $\quad x(t) = (x^1(t), \dots, x^n(t))$

F^{-1}

$$\boxed{\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = \left(\ddot{h}^\alpha + \tilde{\Gamma}^\alpha_{\eta\beta} \dot{h}^\eta \dot{h}^\beta\right) \frac{\partial x^i}{\partial h^\alpha}}$$

$\overset{\shortparallel}{0}$ $\quad$ Homework. $\overset{\shortparallel}{0}$

N. (V, h)

$F : M \to N$ local isometry.

$\gamma \subset M$ $\tilde{\gamma} \subset N$ geodesic

s.t. $F(\gamma) = \tilde{\gamma}$

(2). $\exp_p(v)$ is defined. $\Rightarrow$

$$t \longmapsto \exp_p(tv) = \gamma(1, p, tv)$$
$$= \underline{\gamma(t, p, v)}$$
is a geodesic

F local isometry. (1) $\Rightarrow t \mapsto F(\exp_p(tv))$ is a geodesic on N

$$F(\exp_p(tv))\Big|_{t=0} = F(p)$$

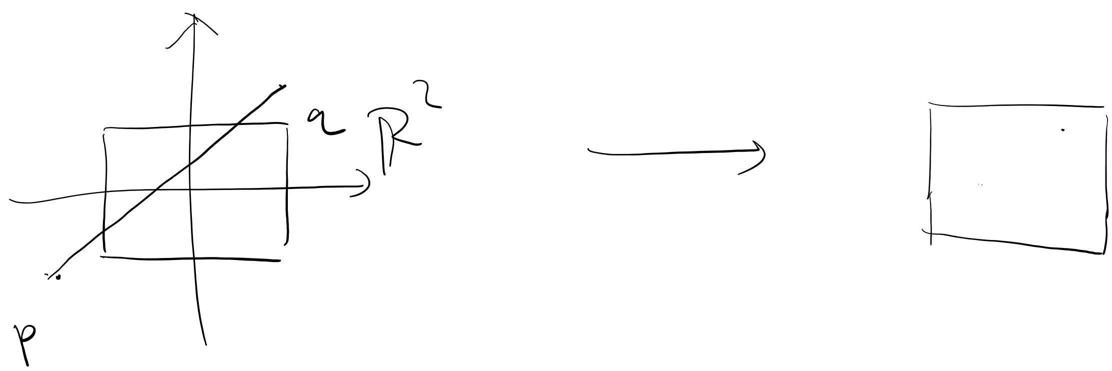
$$\frac{d}{dt}\Big|_{t=0} F(\exp_p(tv)) \overset{\text{def.}}{=} dF(p)\left(\frac{d}{dt}\Big|_{t=0}\exp_p(tv)\right)$$
$$= dF(p)(v)$$

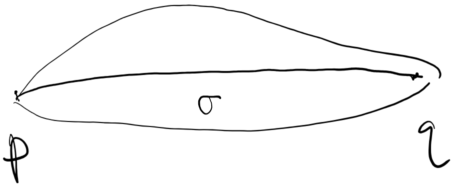
$$\Rightarrow F(\exp_p(tv)) = \exp_{F(p)}(t\,dF(p)(v))$$

let $t = 1$.

(3), (4) ✓ $\square$

Example. $\pi: \mathbb{R}^n \to T^n$ local isometry





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