第八讲

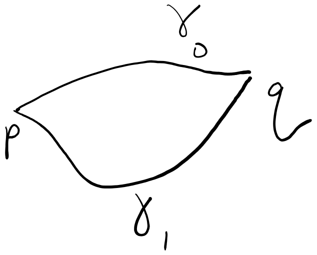
2020年3月19日 13:34

Geodesic: local existence and uniqueness $\Leftarrow$ ODE

Hopf-Rinow: global results

geometry in large

Existence of shortest geodesics in a given homotopy class.



Def: $\gamma_0, \gamma_1 : I = [0,1] \longrightarrow M$

with $\gamma_0(0) = p = \gamma_1(0)$

$\gamma_0(1) = q = \gamma_1(1)$

If $\exists$ continuous map $\Gamma : I \times I \to M$

s.t. $\Gamma(0, s) = p, \quad \forall s \in I,$

$\Gamma(1, s) = q, \quad \forall s \in I$

$\Gamma(t, 0) = \gamma_0(t), \quad \Gamma(t, 1) = \gamma_1(t)$

Then we ~~call~~ say γ_0 and γ_1 are homotopic.

Closed: $c_0, c_1 : S^1 \longrightarrow M$

$[0, 2\pi)$

If $\exists$ continuous map $\Gamma : S^1 \times I \to M$

s.t. $\Gamma(t, 0) = c_0(t), \quad \Gamma(t, 1) = c_1(t)$

then we say c_0, c_1 are homotopic

Aim: Find shortest curve in a given homotopy class.

Lemma: Let M be $\boxed{\text{compact}}$ Rie. mfld.

Let $\rho_0 > 0$ be a number s.t. $\forall p, q \in M$, $d(p,q) < \rho_0$

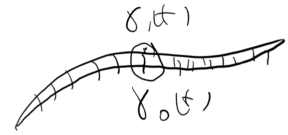
$\Rightarrow \exists\,!$ shortest curve from p to q.

Let $\gamma_0, \gamma_1 : I \longrightarrow M$ be two curves with

(S^1)

Let $\gamma_0, \gamma_1 : \underset{(S^1)}{I} \longrightarrow M$ be two curves with

$$d(\gamma_0(t), \gamma_1(t)) \le \rho_0 \qquad \forall t \in I \ (S^1)$$



Then γ_0, γ_1 are homotopic. $I \to M$

Pf. $\exists !$ shortest curve $s \mapsto C_t(s)$ s.t. $C_t(0) = \gamma_0(t)$, $C_t(1) = \gamma_1(t)$

parametrized prop. to arc length. $t \in I = [0,1]$

$$\Gamma : I \times I \longrightarrow M$$

$$(t, s) \longmapsto \Gamma(t,s) = C_t(s)$$

$$= \boxed{\exp_{\gamma_0(t)}\left(s \exp^{-1}_{\gamma_0(t)} \gamma_1(t)\right)}$$

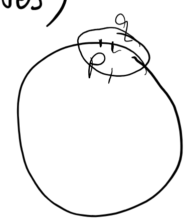
$s \in [0,1] = I$.

□

Prop. γ is a shortest curve in its homotopy class (either ~~with~~ of curves with fixed endpoints or of closed curves)

Then $\exists$ a parametrization s.t. $\gamma = \gamma(t)$ is a geodesic.

$\cancel{\gamma}\ I = [0,1]$



Proof: $\text{Length}(\gamma) \le \text{Length}(\tilde{\gamma})$

parametrized prop. to arc length, where $\tilde{\gamma}$ homotopic to γ s.t. $\tilde{\gamma}(0) = p$, $\tilde{\gamma}(1) = q$.

If γ is not a geodesic, then there exists points p_1, q_1 on γ s.t. length of the arc from p_1 to q_1 is smaller $< \delta/2$ ($B_{p_1}(\delta_{p_1})$ is a ~~totally~~ totally normal neighborhood of p_1), and this arc is not a geodesic. We can then replace the original arc by the unique shortest geodesic from p_1 to q_1

p_1 ~~$B(p_1, \delta$~~ $\overline{B_{p_1}(\delta_{p_1})}$ compt.

$\exists\ \delta > 0$, s.t. $\forall q \in \overline{B_{p_1}(\delta_{p_1})}$.

$B_q(\delta)$ is a normal neighborhood of q.

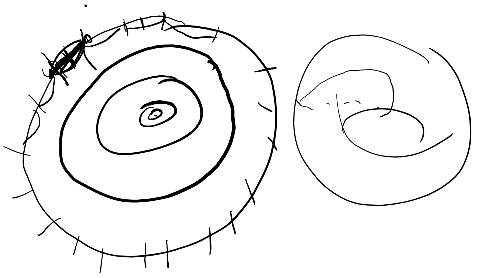
$\Rightarrow$ The "new" curve is homotopic to the original one. $\square$

<u>Theorem</u> 4.1 Let M be a <u>compact</u> Rie mfld.
Then every homotopy class of closed curves in M contains a curve which is a shortest curve in its homotopy class and a geodesic.

<u>Proof</u>:

$\exists!$ shortest geodesic

$\delta > 0$



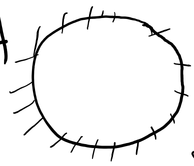
$$\inf_{\gamma \in [\gamma_0]} \text{Length}(\gamma)$$

Choose a minimizing sequence $(\gamma_n)_{n \in \mathbb{N}}$

$$\lim_{n\to\infty} \text{Length}(\gamma_n) = \inf_{\gamma \in [\gamma_0]} \text{Length}(\gamma)$$

For each curve $\gamma_n: S^1 \longrightarrow M$ parametrized proportionally to arc length

$$\exists \ \boxed{0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 2\pi}$$



S^1

s.t. $\text{Length}(\gamma_n |_{[t_{j-1}, t_j]}) < \frac{\delta}{2}$

Replacing $\gamma_n|_{[t_{j-1}, t_j]}$ by the shortest geodesic between $\gamma_n(t_{j-1})$ and $\gamma_n(t_j)$

$\Rightarrow$ · in the same homotopic class.

· length no increasing.

$(\gamma_n)_{n\in\mathbb{N}}$ minimizing sequence,

$\sup_n \{\text{Length}(\gamma_n)\} < \infty \Rightarrow$ We can choose m to be a uniform constant.

γ_1 : $p_{0,1}$, $p_{1,1}$, ..., $p_{m,1}$

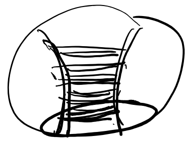
γ_2 : $p_{0,2}$, $p_{1,2}$, ..., $p_{m,2}$

$\vdots$

γ_n : $p_{0,n}$, $p_{1,n}$, ..., $p_{m,n}$

$\vdots$

M compact "diagonal argument"



$\exists$ subsequence,

$p_0, p_1, \ldots, p_m \longrightarrow \gamma$

$\forall j$, $\underline{p_{j-1,n}, p_{j,n}}$ $\exists!$ shortest geodesic

$$\underline{d(p_0, p_1) = \lim_{j\to\infty} d(p_{0,j}, p_{1,j})} \le \delta/2 < \delta$$

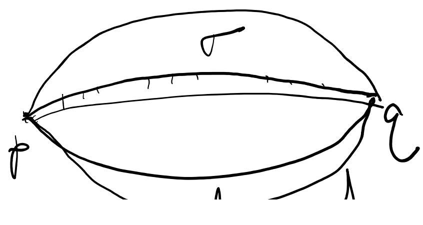
γ piecewise geodesic closed.

in the same homotopy class as γ_n.

$$\text{Length}(\gamma) = \lim_{n\to\infty} \text{Length}(\gamma_n) = \inf_{\gamma\in[\gamma_0]} \text{Length}(\gamma).$$

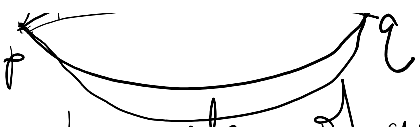
$\square$

① $\underline{M \text{ compact}}$.



<u>Thm 4.2</u> Let M be a <u>complete</u>

Thm 4.2 Let M be a complete Rie. mfld, $p, q \in M$. Every homotopy class of curves from p to q contains a geodesic γ that minimizing length among all curves in the same homotopy class.



· If M is simply connected, Hopf-Rinow.

$(\tilde{M}, \pi^* g)$ universal covering $\tilde{p} - \tilde{q}$

$\downarrow \pi$

(M, g) complete $\quad p, q$

C^∞ mfld.

Recall: A covering map: $\pi : \tilde{M} \longrightarrow M$

$\pi^{-1}(U) \longleftarrow U$

is a continuous map which is surjective, such $\forall p \in M$,

$\exists U \ni p$, is evenly covered:

every connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U.

$\Updownarrow$

diffeomorphically $\rightarrow$ A smooth covering map.

$\Updownarrow$ isometry

Any Rie metric on M induces a Rie. metric on $\tilde{M}$

π: a Riemannian covering map.

π is a local isometry.

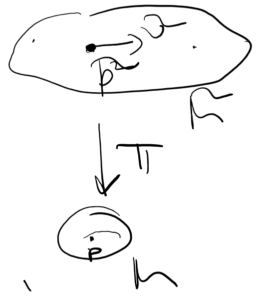
Lemma: $\tilde{M}$, M, Rie mflds.

$\pi : \tilde{M} \to M$ is Riemannian covering map.

Then if M is complete, then $\tilde{M}$ is also complete.

Then | if M is complete, then $\tilde{M}$ is also complete.

M is complete iff $\tilde{M}$ is complete



Proof: $\Rightarrow$ Let $\forall\ \tilde{p} \in \tilde{M}$, $\forall\ \tilde{v} \in T_{\tilde{p}}\tilde{M}$

$p = \pi(\tilde{p})$, $v = d\pi(\tilde{p})(\tilde{v}) \in T_pM$

M cpl. $\Rightarrow \exists$ a geodesic γ s.t. $\gamma(0) = p$, $\dot{\gamma}(0) = v$ is defined on $[0, \infty)$

| Path Lifting |

$\forall\ \gamma : I \to M$ contin. $\exists\ \tilde{\gamma} : I \to \tilde{M}$ s.t.

$\pi \circ \tilde{\gamma} = \gamma$

$\Rightarrow \exists!$ Lift of γ, $\tilde{\gamma}$ s.t. $\tilde{\gamma}(0) = \tilde{p}$

$\pi \circ \tilde{\gamma} = \gamma \Rightarrow \dot{\tilde{\gamma}}(0) = \tilde{v}$.

and $\tilde{\gamma}$ is defined on $[0, \infty)$

$\xRightarrow{\text{Hopf-Rinow}}$ $\tilde{M}$ complete.

"$\Leftarrow$" $\forall\ p \in M$, $\forall\ v \in T_pM$.

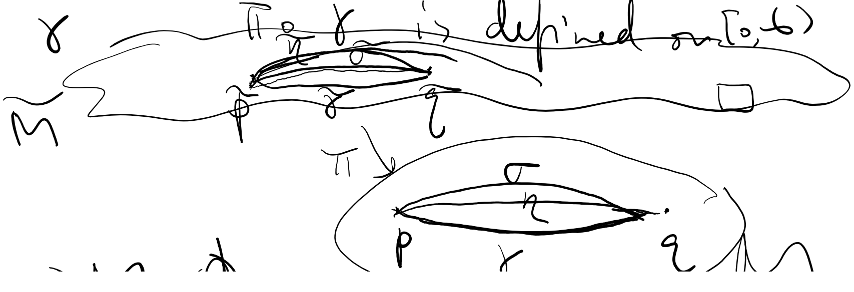
$\exists\ \gamma : I \to M$ geodesic $\gamma(0) = p$, $\dot{\gamma}(0) = v$

Let $\tilde{\gamma}$ be a lift of γ in $\tilde{M}$ starting at $\tilde{p}$

π local isometry $\Rightarrow \tilde{\gamma}$ is a geodesic $\tilde{M} \xrightarrow{\pi} M$

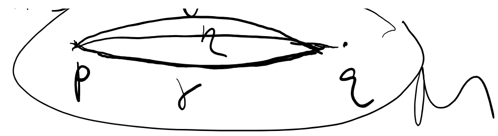
$\tilde{M}$ complete $\Rightarrow \tilde{\gamma}$ is defined $[0, \infty)$

$\pi \circ \tilde{\gamma}\big|_I = \gamma$ ~~$\pi \circ \tilde{\gamma}$ is defined on $[0,\infty)$~~



Proof of 4.2.

Proof of T.C.



Consider $\pi : \widetilde{M} \to M$ universal covering map. Riemannian

$\sigma : [0,1] \to M$. from p to q $\quad [\sigma]$

Choose $\tilde{p} \in \pi^{-1}(p)$, and $\tilde{\sigma}$ be the lift of σ starting at $\tilde{p}$ $\quad \tilde{\sigma} : [0,1] \to \widetilde{M} \quad \pi \circ \tilde{\sigma} = \sigma$

denote $\tilde{q} =: \tilde{\sigma}(1)$

In particular, $\pi(\tilde{q}) = \pi \circ \tilde{\sigma}(1) = \sigma(1) = q$.

$\widetilde{M}$ cpl. $\Rightarrow \exists$ shortest geodesic $\tilde{\gamma}$ from $\tilde{p}$ to $\tilde{q}$.

π. local isometry $\gamma := \pi \circ \tilde{\gamma}$ is a geodesic in M. from p to q

① $\tilde{\sigma} : [0,1] \to \widetilde{M} \quad \tilde{\sigma}(0) = p,\ \tilde{\sigma}(1) = q$

$\tilde{\gamma} : [0,1] \to \widetilde{M} : \tilde{\gamma}(0) = p,\ \tilde{\gamma}(1) = q$

$\tilde{\sigma}$ is homotopic to $\tilde{\gamma}$ in $\widetilde{M}$

$\pi \circ \tilde{\gamma} = \gamma \quad , \pi \circ \tilde{\sigma} = \sigma \quad \Rightarrow \gamma$ and σ is homotopic.

② If $\underline{\eta \in [\sigma] : I \to M}$, $\underline{\text{Aim}}$ Length$(\gamma) \leq$ Length(η)

$\pi \circ \tilde{\gamma} = \gamma \quad \tilde{\gamma}(0) = \tilde{p}$

$\exists!$ lift $\tilde{\eta}$ of η starting at $\tilde{p}$

$\tilde{\eta}$ ends at $\tilde{q}$. (Monodromy theorem)

$\underline{\Gamma : I \times I \to M}$ continuous

$\Gamma(0,0) = p.$

$$\begin{array}{ccc} \mathrm{Length}(\tilde{\eta}) & \geq & \mathrm{Length}(\tilde{\gamma}) \\ \| & & \| \\ \mathrm{Length}(\eta) & \geq & \mathrm{Length}(\gamma) \end{array} \Rightarrow \gamma \text{ is a shortest one}$$

$\square$

<u>Thm</u>: (M, g), $(\tilde{M}, \tilde{g})$

$\tilde{M}$ complete, $\pi: \tilde{M} \to M$ ~~Rie covering map~~ <u>local isometry</u>

$\Rightarrow$ M complete, $\pi: \tilde{M} \to M$ Rie. covering map.

<u>Prof</u> based on local existence and uniqueness of geodesic.

$\square$

下课.