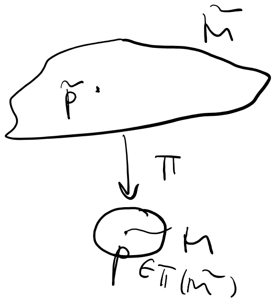
<u>Thm</u>. Let $(\tilde{M}, \tilde{g})$ and (M, g) be two Rie. mflds. $(\tilde{M}, \tilde{g})$ is complete. Let $\pi : \tilde{M} \longrightarrow M$ is a local isometry. Then (M, g) is complete and π is a Rie. covering map.

<u>Proof</u>: · Path-lifting property for geodesics.



Let $p \in \underline{\pi(\tilde{M})}$, $\tilde{p} \in \pi^{-1}(p)$

Let $\gamma : I = [0,1] \longrightarrow M$ be a geodesic

with $\gamma(0) = p$, $\dot{\gamma}(0) = v$

π local isometry $\Rightarrow$ $d\pi(\tilde{p}) : T_{\tilde{p}}\tilde{M} \longrightarrow T_p M$
linear isometry
$v \in T_p M$

$$\boxed{\tilde{v} := \left(d\pi(\tilde{p})\right)^{-1}(v) \in T_{\tilde{p}}\tilde{M}}$$

$\exists$ a geodesic $\tilde{\gamma}$ on $\tilde{M}$ with $\tilde{\gamma}(0) = \tilde{p}$, $\dot{\tilde{\gamma}}(0) = \tilde{v}$
$\Uparrow$
complete. $\tilde{\gamma}$ is defined on $[0, \infty)$

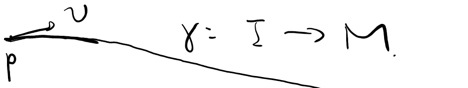
$\pi \circ \tilde{\gamma} : [0, \infty) \to M$ is a geodesic

$\pi \circ \tilde{\gamma}(0) = \pi(\tilde{p}) = p$,

$(\pi \circ \tilde{\gamma})^{\cdot}(0) = d\pi_{(\tilde{\gamma}(0))}(\dot{\tilde{\gamma}}(0)) = d\pi(\tilde{p})(\tilde{v}) = v$

By uniquen of geodesics, we know $\pi \circ (\tilde{\gamma}|_I) = \underline{\gamma}$.

· M <u>complete</u>.



$\gamma : I \to M$. $\pi \circ \tilde{\gamma} : [0, \infty) \to M$

<u>Remains</u> to show π is a Rie covering map.

① π is surjective.

② $\forall p \in M$, $\exists p \in U$ that is <u>evenly covered</u>.

$$\pi^{-1}(U) = \bigcup_{\alpha \in A} \tilde{U}_\alpha, \quad \tilde{U}_\alpha \cap \tilde{U}_\beta = \phi$$

$\pi : U_\alpha \to U$ diffeomorphism

<u>① π is surjective</u>. $\pi : \tilde{M} \to M$

$\forall \tilde{p} \in \tilde{M}$, $\underline{p} = \pi(\tilde{p}) \in M$. $p \in \pi(\tilde{M})$

$\forall q \in M$ [illegible] $q \in \pi(\tilde{M})$

$\forall \tilde{p} \in \tilde{M}$, $p = \pi(\tilde{p}) \in M$. $p \in \pi(\tilde{M})$.

$\forall q \in M$, $\overset{?}{\Rightarrow}$ $q \in \pi(\tilde{M})$.

M cpl. $\Rightarrow$ $\exists$ a shortest geodesic γ from p to q.
$\gamma : [0, r] \to M$.

Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{p}$, $\underline{\pi \circ \tilde{\gamma} = \gamma}$

denote $d(p, q) = r$, $q = \gamma(r)$

$$\pi\big(\tilde{\gamma}(r)\big) = \gamma(r) = q. \Rightarrow q \in \pi(\tilde{M}).$$

② Let $p \in M$. $U = B_p(\varepsilon)$ is a normal ball.

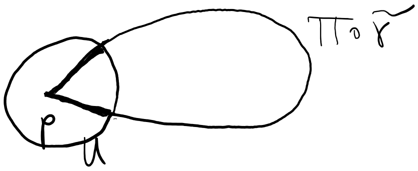
$$\pi^{-1}(p) = (\tilde{p}_\alpha)_{\alpha \in A}$$

$\tilde{U}_\alpha$ = metric ball of radius ε around $\tilde{p}_\alpha$.

<u>Claim</u> $\tilde{U}_\alpha \cap \tilde{U}_\beta = \emptyset$? $\forall \alpha \neq \beta$ $\Leftarrow$ $\underline{d(\tilde{p}_\alpha, \tilde{p}_\beta) \geq 2\varepsilon}$

<u>Proof</u>: $\tilde{M}$ cpl. $\Rightarrow$ $\exists$ a shortest geodesic $\tilde{\gamma}$ from $\tilde{p}_\alpha$ to $\tilde{p}_\beta$

π local isometry $\pi \circ \tilde{\gamma}$ is a geodesic in M from p to p



$p \in U = B_p(\varepsilon)$

$$\underset{\text{Length}(\tilde{\gamma}) = d(\tilde{p}_\alpha, \tilde{p}_\beta)}{\underset{\shortparallel}{\text{Length}(\pi \circ \tilde{\gamma})}} \geq 2\varepsilon$$

$\square$

<u>Claim</u>: $\pi^{-1}(U) = \bigcup_{\alpha \in A} \tilde{U}_\alpha$ $\Longleftrightarrow$ $\pi\left(\bigcup_{\alpha \in A} \tilde{U}_\alpha\right) = U$

<u>Proof</u>: $\bigcup_{\alpha \in A} \tilde{U}_\alpha \subset \pi^{-1}(U)$

$\forall \tilde{q} \in \bigcup_{\alpha \in A} \tilde{U}_\alpha$, $\underline{\tilde{q} \in \tilde{U}_\alpha}$ for some α. $\tilde{U}_\alpha = B_{\tilde{p}_\alpha}(\varepsilon)$

? $\tilde{q} \in \pi^{-1}(U) \Leftrightarrow \underline{\pi(\tilde{q}) \in U = B_p(\varepsilon)}$

$\tilde{d}(\tilde{q}, \tilde{p}_\alpha) < \varepsilon$, $\tilde{M}$ cpl. $\Rightarrow$ $\exists$ shortest geodesic $\tilde{\gamma}$ from $\tilde{p}_\alpha$ to $\tilde{q}$

$\underline{\pi \circ \tilde{\gamma}}$ is a geodesic on M from p to $q := \pi(\tilde{q})$

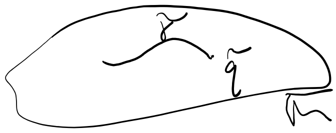
$$\text{Length}(\pi \circ \tilde{\gamma}) = \text{Length}(\tilde{\gamma}) < \varepsilon$$

$$\Rightarrow d(p, q) < \varepsilon. \Rightarrow q = \pi(\tilde{q}) \in U.$$

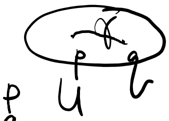
. $\pi^{-1}(U) \subset \bigcup \tilde{U}_\alpha$

$d(p,q) < \varepsilon$, $\tilde{q} \ \pi(\tilde{q}) \in U$.

- $\pi^{-1}(U) \subset \underline{\bigcup_{\alpha \in A} \tilde{U}_\alpha}$



$\forall\ \underline{\tilde{q}} \in \pi^{-1}(U)$, set $\underline{q := \pi(\tilde{q}) \in U}$.



$\exists$ a shortest geodesic γ in U from q to p.

$\gamma(0) = q$, $\gamma(r) = p$, $r = d(p,q) \le \varepsilon$

Let $\tilde{\gamma}$ be the <u>lift</u> of γ in $\tilde{M}$ starting at $\tilde{q}$

$\pi \circ \tilde{\gamma} = \gamma \Rightarrow \pi(\tilde{\gamma}(r)) = \gamma(r) = p$

$\Rightarrow \underline{\tilde{\gamma}(r)} \in \pi^{-1}(p) = \bigcup_{\alpha \in A} \tilde{p}_\alpha$

$\Rightarrow \tilde{\gamma}(r) = \tilde{p}_\alpha$ for some $\alpha \in A$.

$\Rightarrow \tilde{d}(\tilde{q}, \tilde{p}_\alpha) < \varepsilon \Rightarrow \tilde{q} \in B_{\tilde{p}_\alpha}(\varepsilon) = \tilde{U}_\alpha$. $\square$

- $\pi : \underline{\tilde{U}_\alpha} \longrightarrow U$ $\quad \pi(\tilde{U}_\alpha) = U$

  is a diffeomorphism.

① π injective ② $\underline{d\pi \text{ is nonsingular everywhere}}$

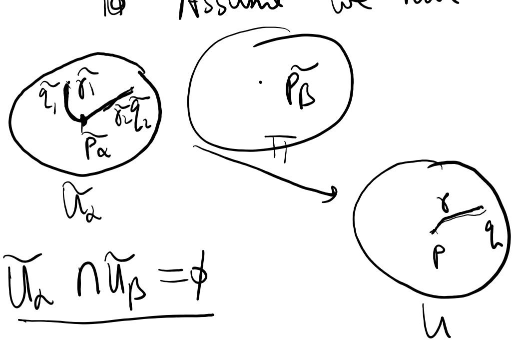
$\Uparrow$

π is a local isometry.

lift

$\underline{\pi(\tilde{\gamma}(r)) = q}$ $\quad r = d(p,q)$ $\quad \Rightarrow \pi$ surjective.

① Assume we have $\tilde{q}_1, \tilde{q}_2 \in \tilde{U}_\alpha$ st. $\pi(\tilde{q}_1) = \pi(\tilde{q}_2)$
$:= q \in U$.



$\underline{\tilde{U}_\alpha \cap \tilde{U}_\beta = \phi}$

$\gamma : [0,1] \to U$

$\gamma(0) = q$, $\gamma(1) = p$

<u>lift of γ</u> : $\tilde{\gamma}_1 : [0,1] \to U$

$\tilde{\gamma}_1(0) = \tilde{q}_1$

$\tilde{\gamma}_2 : [0,1] \to U \quad \tilde{\gamma}_2(0) = \tilde{q}_2$

$\underline{\pi \circ \tilde{\gamma}_i} = \underline{\gamma}$, $i = 1,2$.

$\pi(\tilde{\gamma}_i(1)) = \gamma(1) = p \Rightarrow \tilde{\gamma}_1(1) = \tilde{\gamma}_2(1) = \tilde{p}_\alpha$

$d\pi(\tilde{p}_\alpha)(\dot{\gamma}_i(1)) = \dot{\gamma}(1)$, $i = 1,2$.

contracting to the fact that π is a local isometry $\square$

休息到 10:48.

(II ½) Function Spaces and Laplace-Beltrami operator on Rie. mfld.

Geodesic curves $\gamma : I \subset \mathbb{R} \to M$ map.

$\hookrightarrow$ critical point of Energy functional.

functions $f : M \longrightarrow \mathbb{R}$

harmonic fct $\to$ critical point of Energy functional

(M, g) Rie mfld. regular Borel measure vol_g

$\forall K \subset M$ cpt $vol_g(K) < \infty$

Definition: If $0 < p < \infty$, if $f : M \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function on M, define

$$\|f\|_p := \left(\int_M |f|^p \, dvol_g \right)^{1/p}$$

The L^p-space is

$$L^p(vol_g) = \left\{ f : M \to \mathbb{R} \cup \{\pm\infty\} : \begin{array}{l} f \text{ is measurable} \\ \|f\|_p < \infty \end{array} \right\}$$

- $L^p(vol_g)$ is complete for $1 \le p < \infty$
- $C^0_0(M)$ is dense in $L^p(vol_g)$, $1 \le p < \infty$

$L^p(vol_g)$ as the completion of $C^0_0(M)$ w.r.t. $\|\cdot\|_p$, $1 \le p < \infty$

In particular, $p = 2$, $L^2(vol_g)$

inner product, $(f_1, f_2) := \int_M f_1 \cdot f_2 \, dvol_g$, $\forall f_1, f_2 \in L^2(vol_g)$

$L^2(vol_g)$ is Hilbert space.

<u>Aim</u>. $L^2(\mathrm{vol}_g)$, ((M,g) <u>compact</u>) has a complete orthonormal basis given by the eigenfunction of the Laplace-Beltrami operator.

§1. Gradient, divergence, Laplace-Beltrami operator.

$(\mathbb{R}^n, g_{eucl})$ $\qquad f: \mathbb{R}^n \to \mathbb{R}$ $\quad C^\infty$

gradient of f: $\nabla f =: \mathrm{grad} f$: is smooth vector field.

$$\nabla f := \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i} \cong \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)$$

For a smooth vector field $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, the divergence

$$\mathrm{div}(X) := \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}$$

<u>Laplace operator</u>. Δ:

$$\forall f \in C^\infty(\mathbb{R}^n), \qquad \Delta f := \sum_{i=1}^n \frac{\partial^2 f}{(\partial x^i)^2} = \mathrm{div}(\nabla f).$$

(M,g)

· Gradient. $\forall f \in C^\infty(M)$

The gradient $\nabla f := \mathrm{grad} f$ of f is a smooth vector field. s.t.

$$\langle \mathrm{grad} f, X \rangle = X(f), \ \forall \text{ smooth v. f. } X.$$

<u>Expression in local chart</u>: (U, x)

$\forall X = X^i \frac{\partial}{\partial x^i}$, we have

$$\langle \mathrm{grad} f, X \rangle = \langle \underbrace{(\mathrm{grad} f)^j}_{} \frac{\partial}{\partial x^j}, X^i \frac{\partial}{\partial x^i} \rangle$$

$$= (\mathrm{grad} f)^j X^i g_{ji}$$

$$\| $$

$$X(f) = X^i \frac{\partial f}{\partial x^i}, \ \forall X^1, \dots, X^n$$

$$\Leftrightarrow (\mathrm{grad} f)^j X^i g_{ji} = X^i \frac{\partial f}{\partial x^i}, \ \underline{\forall X^1, \dots, X^n}.$$

$$\Rightarrow (\mathrm{grad} f)^j \underbrace{g_{ji} g^{ik}}_{\delta_j^k} = \frac{\partial f}{\partial x^i} g^{ik}$$

δ_j^k

$$\Rightarrow \quad (\mathrm{grad} f)^k = g^{ik} \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \quad \mathrm{grad} f = \sum_{j=1}^{n} (\mathrm{grad} f)^j \frac{\partial}{\partial x^j} = \underline{\sum_{j=1}^{n} \sum_{i=1}^{n}} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

$$\boxed{\mathrm{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}}$$

Remark (1) $\left(\mathbb{R}^n, \ (g_{ij}) = (\delta_{ij})\right)$ ✓

(2) $f \in C^\infty(M)$, $\underline{f^{-1}(c)}$ $c \in \mathbb{R}$

c regular value of f, $f^{-1}(c)$ is a submfld.

grad f vertical to $\underline{f^{-1}(c)}$

$\forall \ X \in \Gamma(T f^{-1}(c))$

$$\langle \mathrm{grad} f, X \rangle \Big|_{f^{-1}(c)} = X(f) \Big|_{f^{-1}(c)} = 0. \quad \square$$

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· Divergence: smooth vector fields X on (M^n, g)

$\mathrm{div}(X)$ chart (U, x)

volume element: volume n-form:

$$\Omega_0 := \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n.$$

$G := \det(g_{ij})$ $\quad \Omega_0 = \sqrt{G} \, dx^1 \wedge \cdots \wedge dx^n$

Lie derivative $\boxed{\mathcal{L}_X \Omega_0 = \mathrm{div}(X) \, \Omega_0}$ define.

linear space of n-form on $x(U) \subset \mathbb{R}^n$ is of dimension one.

· M orientable, $\Rightarrow$ global volume n-form.

Remark (1). $\underline{\mathrm{div}(X) \text{ is well defined}}$.

Remark (1). $\underline{\mathrm{div}(X) \text{ is well defined}}$.

(2). (U, x) chart.

$\underline{\text{Cartan's magical formula}}$: M. ω p-form on M.

X vector field.

$$\mathcal{L}_X \omega = i(X)\, d\omega + d\big(i(X)\omega\big)$$

$$\underline{i(X)\omega}\,(Y_1, \dots, Y_{p-1}) := \omega(X, Y_1, \dots, Y_{p-1})$$

Apply Cartan's formula to Ω_0

$$\mathcal{L}_X \Omega_0 = i(X)\, \underbrace{d\Omega_0}_{=0} + d(i(X)\Omega_0)$$

$$= d(i(X)\Omega_0)$$

$$\Rightarrow \quad \boxed{d(i(X)\Omega_0) = \mathrm{div}(X)\,\Omega_0}$$

$X = X^i \frac{\partial}{\partial x^i}$ $\quad i(X)\Omega_0 = i(X)\sqrt{G}\, dx^1 \wedge \dots \wedge dx^n = X^i \sqrt{G} \cdot \underbrace{i\left(\frac{\partial}{\partial x^i}\right) dx^1 \wedge \dots \wedge dx^n}$

$\forall Y_1, \dots, Y_{n-1}$,

$$i\left(\frac{\partial}{\partial x^i}\right) dx^1 \wedge \dots \wedge dx^n \,(Y_1, \dots, Y_{n-1})$$

$$= \underline{dx^1 \wedge \dots \wedge dx^n}\left(\frac{\partial}{\partial x^i}, Y_1, \dots, Y_{n-1}\right)$$

$$\doteq (-1)^{i-1}\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n \,(Y_1, \dots, Y_{n-1})$$

$$\Rightarrow \quad i(X)\Omega_0 = \sum_{i=1}^{n} X^i \sqrt{G}\, (-1)^{i-1} dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

$$d(i(X)\Omega_0) = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial (X^i \sqrt{G})}{\partial x^k}\, \underline{dx^k \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial (X^i \sqrt{G})}{\partial x^i}\, \underline{dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n}$$

$$= \underbrace{\frac{1}{\sqrt{G}} \sum_{i=1}^{n} \frac{\partial (X^i \sqrt{G})}{\partial x^i}}\, \sqrt{G}\, \underline{dx^1 \wedge \dots \wedge dx^n}$$

$$= (\mathrm{div}(X))\, \Omega_0$$

$$\Rightarrow \quad \boxed{\mathrm{div}(X) = \frac{1}{\sqrt{G}} \sum_{i=1}^{n} \frac{\partial (\sqrt{G}\, X^i)}{\partial x^i}}, \quad X = X^i \frac{\partial}{\partial x^i}$$

Remark. $(\mathbb{R}^n, (\delta_{ij}))$

Thm (Divergence Thm) Let X be a smooth vector field on (M, g) with compact support. Then

$$\int_M \operatorname{div}(X)\, dvol_g = 0.$$

Proof: partition of unity.

$\{U_\alpha\}_{\alpha\in A}$ locally finite covering via charts (U_α, x_α)

$\{\phi_\alpha\}_{\alpha\in A}$ p.o.u. subordinated to $\{U_\alpha\}_{\alpha\in A}$.

$\#\{\alpha\in A \mid \operatorname{Supp}(X)\cap U_\alpha\} < \infty$

$$X = \sum_{\alpha\in A} \phi_\alpha X$$

$$\int_M \operatorname{div}(X)\, dvol_g = \int_M \operatorname{div}\Big(\sum_\alpha \phi_\alpha X\Big)\, dvol_g$$

$$= \int_{\operatorname{supp}(X)} \sum_\alpha \operatorname{div}(\phi_\alpha X)\, dvol_g = \sum_\alpha \underbrace{\int_{U_\alpha} \operatorname{div}(\phi_\alpha X)\, dvol_g}_{=0}$$

$$\int_{U_\alpha} \operatorname{div}(\phi_\alpha X)\, dvol_g$$

$$= \int_{x_\alpha(U_\alpha)} \frac{1}{\cancel{\sqrt{G}}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\,\phi_\alpha X^i\right) \cancel{\sqrt{G}}\, dx^1\cdots dx^n$$

$$= \int_{x_\alpha(U_\alpha)} \sum_{i=1}^n \underbrace{\frac{\partial}{\partial x^i}\left(\sqrt{G}\,\phi_\alpha X^i\right)}_{\in C_0^\infty(U_\alpha)} dx^1\cdots dx^n = 0.$$

□

Laplace - Beltrami operator : Δ.

$\forall f \in C^\infty(M)$, $\Delta f := \operatorname{div}(\operatorname{grad} f)$

in local chart (U, x):

$$\Delta f := \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\Big(\sqrt{G}\cdot \underbrace{(\operatorname{grad} f)^i}_{g^{ij}\frac{\partial f}{\partial x^j}}\Big)$$

$$\boxed{\Delta f := \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ij} \frac{\partial f}{\partial x^j}\right)}.$$

$f \in C_0^\infty \Rightarrow \int_M \Delta f \, dvol_g = 0.$

- <u>Green's formula</u>:

<u>Thm</u>: Let $f, h \in C^\infty(M)$, at least one of them has cpt support. Then

$$\int_M f \Delta h \, dvol_g = -\int_M \langle \operatorname{grad} f, \operatorname{grad} h \rangle \, dvol_g$$
$$= \int_M \Delta f \cdot h \, dvol_g .$$

in short: $\underline{(f, \Delta h) = -(\operatorname{grad} f, \operatorname{grad} h)} = (\Delta f, h)$

<u>Proof</u>: $\cdot \; \boxed{\operatorname{div}(fX) = f \cdot \operatorname{div}(X) + \langle \operatorname{grad} f, X \rangle}$

Apply the above formula to $X = \operatorname{grad} h$

$$\underbrace{\int_M \operatorname{div}(f \cdot \operatorname{grad} h)}_{= 0} = \int_M f \Delta h + \int_M \langle \operatorname{grad} f, \operatorname{grad} h \rangle$$

$$(f, \Delta h) = (\Delta f, h) . \qquad \square$$

<u>Remark</u>: (M, g) compact. $\forall f, h \in C^\infty(M)$,
$$(f, \Delta h) = (\Delta f, h)$$

- Energy functional. $f: M \to \mathbb{R}$ C^∞

$$E(f) := \frac{1}{2} \int_M \langle \nabla f, \nabla f \rangle \, dvol_g$$

<u>Thm</u>: A smooth critical point f of E in the sense

$$\underline{\left.\frac{d}{dt}\right|_{t=0} E(f + t\eta) = 0}$$

for all $\eta \in C_0^\infty(M)$, is harmonic, i.e., $\Delta f = 0$.

<u>Proof</u>: We compute $\langle \nabla f + t \nabla \eta, \nabla f + t \nabla \eta \rangle$

$$0 = \underline{\left.\frac{d}{dt}\right|_{t=0}} \frac{1}{2} \int_M \langle \nabla (f + t\eta), \nabla (f + t\eta) \rangle \, dvol_g$$

$$= \frac{1}{2} \int \frac{d}{dt}\Big| \, 2t \langle \nabla f, \nabla \eta \rangle \, dvol_g$$

$$= \frac{1}{2}\int_M \frac{d}{dt}\Big|_{t=0} 2t \langle \nabla f, \nabla \eta \rangle \, dvol_g$$

$$= \int_M \langle \nabla f, \nabla \eta \rangle \, dvol_g$$

$$\overset{\text{Green's}}{=} -\int_M \Delta f \cdot \eta \, dvol_g . \qquad \forall \eta \in C_0^\infty(M).$$

$\Rightarrow$ $\underline{\Delta f = 0}$ harmonicity. $\square$

harmonic map.

下课.