第二十二讲

2020年5月10日 0:10

Synge Theorem : 1936

On the connectivity of spaces of positive curvature.
(Quarterly Journal of Math. (Oxford Series), 7, 316-320)

Thm (Synge) (M^n, g), $\sec > 0$, M compact.

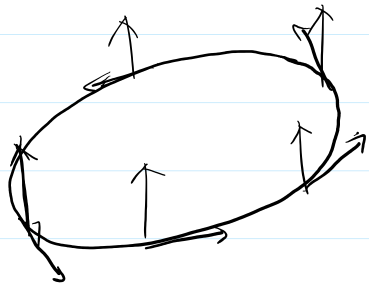
(a) If M is orientable, n is even, then $\pi_1(M) = \{1\}$, i.e. M is simply connected

(b) If M is nonorientable, n is even, then $\pi_1(M) = \mathbb{Z}_2$

(c) If M is nonorientable, n is odd, $\boxed{\text{contradiction}}$.
If M is odd dim'l (i.e., n is odd), then M is orientable.

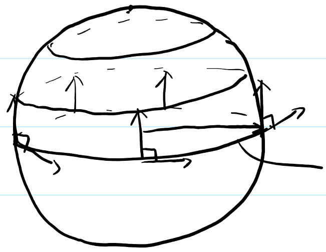
(M, g), $\sec > 0$

Principle : Let $\gamma : [a,b] \to M$ a closed ~~curve~~ geodesic
$\gamma(a) = \gamma(b)$, $\dot\gamma(a) = \dot\gamma(b)$ $\quad E(t)$
If there exists a parallel vector field along γ,
s.t. $\langle E(t), T(t) \rangle = 0$, $\forall t$
Then γ is not minimizing in its free homotopy class.



Proof. A direct application of SVF.

$$E''(0) = \int_a^b \Big(\underbrace{\langle \nabla_T V, \nabla_T V \rangle}_{=0} - \underbrace{\langle R(V, T)T, V \rangle}_{>0} \Big)\, dt < 0. \quad \square$$



S^2

$p = \gamma(a) = \gamma(b)$

Consider : $P := P_{\gamma, a, b} : T_{\gamma(a)} M \longrightarrow T_{\gamma(b)} M$

$$v \longmapsto P_{\gamma, a, b}(v)$$

$T(t)$ is parallel along γ, i.e. $P(T(a)) = T(b) = T(a)$

multiplicity of the eigenvalue $+1$ ≥ 2

Lemma: $\forall\ v, w \in \underline{T_pM}$, $\langle P(v), P(w)\rangle = \langle v, w\rangle$
orthogonal.

matrix $P^T P = Id \Rightarrow \det(P) = \det(P) = \pm 1$.

$Pv = \lambda v \Rightarrow |v|^2 = |\lambda|^2 |v|^2 \Rightarrow \underline{|\lambda| = 1}$

~~eigenvectors~~ : eigenvalues : n

$$\overbrace{\underbrace{\lambda_1, \bar\lambda_1, \dots, \lambda_j, \bar\lambda_j}, \underbrace{-1, \dots, -1}_{k}, \underbrace{+1, \dots, +1}_{l}}^{n} \qquad l \geq 1^{+}$$

Aim. $l \geq 2$.

Lemma: $\boxed{\cdot \text{ If } \boxed{\det(P) = +1},\ n \text{ even, then } l \geq 2}$

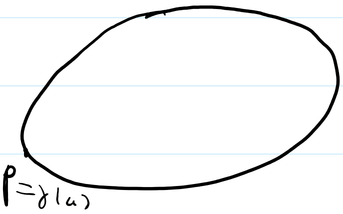
$\cdot$ If $\det(P) = -1$, n odd, then $l \geq 2$.

Proof: $\det(P) = +1 \Rightarrow \left.\begin{array}{l} k \text{ even} \\ n \text{ even} \end{array}\right\} \Rightarrow \left.\begin{array}{l} l \text{ even} \\ l \geq 1 \end{array}\right\} \Rightarrow l \geq 2$.

$\det(P) = -1 \Rightarrow \left.\begin{array}{l} k \text{ odd} \\ n \text{ odd} \end{array}\right\} \Rightarrow \left.\begin{array}{l} l \text{ even} \\ l \geq 1 \end{array}\right\} \Rightarrow l \geq 2$. $\square$

Lemma: Let (M^n, g) be an $\boxed{\text{orientable}}$ Rie. mfld.
$\gamma : [a,b] \to M$ be a C^∞ closed curve. $p = \gamma(a) = \gamma(b)$
Then $P := P_{\gamma,a,b} : T_{\gamma(a)}M \to T_pM$ has determinant $+1$.

Proof; Remains to show $\underline{\det(P) > 0}$

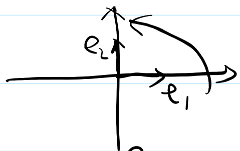


Orientation:

$\mathbb{R}$

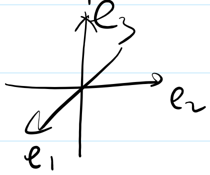
~~e_1~~ (

$\mathbb{R}^2$



$\{e_1, e_2\}$ counterclockwise.
$\{e_2, e_1\}$

$\mathbb{R}^3$



right hand

Vector space V $\quad \{f_1, \dots, f_n\} \quad \{e_1, \dots, e_n\}$

$f_i = a_i^j e_j \qquad \Omega \in \overset{n\text{-covector}}{\Lambda^n(V^*)} \qquad \Omega(f_1, \dots, f_n)$

$$f_j = a_j^i e_i \qquad \Omega \in \underset{\circ}{\Lambda^n(V^*)} \;(n\text{-covector}) \qquad \Omega(f_1,\dots,f_n) = \det(a)\,\Omega(e_1,\dots,e_n)$$

$\det(a_j^i) \begin{array}{l} >0 \\ <0 \end{array} \rightsquigarrow$ equivalence class of orientations.

$\boxed{\text{pointwise orientation.}}$

local

a frame

orientable.

$\exists$ C^∞ n-form, nowhere vanishing, ω $\Leftrightarrow$ orientable)

$\{e_1(a), \dots, e_n(a)\} \subset T_pM$ orthonormal

$P(e_i(a)) = e_i(b) \in T_pM.$

$P_{\gamma;a,t}(e_i(a)) = e_i(t) \in T_{\gamma(t)}M.$

$\omega(e_1(t), e_2(t), \dots, e_n(t))$ a C^∞ fct along γ.

$\underbrace{\omega(e_1(a), \dots, e_n(a)) > 0} \Rightarrow \underbrace{\omega(e_1(b), \dots, e_n(b)) > 0}$

$\Rightarrow \det(P) > 0$ $\square$

Proof. Synge (a). $\left\{\begin{array}{l} \text{orientable} \Rightarrow \det(P) = +1 \\ n \text{ even} \end{array}\right\} \xRightarrow{\text{Linear Algebra}}$ $\boxed{l \geq 2}$

compact $\xRightarrow{\text{Chap 2.}}$ $\exists$ minimizing closed geodesic γ in every homotopy class.

$\left.\text{Sec} > 0\right\} \xRightarrow{\text{SVF}}$ If γ is not a point, γ is a closed geodesic then γ can not be minimizing in its homotopy class.

$\square$

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Proof Synge (c). (M^n, g) sec>0, compact, n odd, $\boxed{\text{non orientable}}$ $\to$ contradiction.

Proof: non-orientable $\Rightarrow$ $\exists$ closed curve c, s.t. [C] is nontrivial and ~~another~~

Proof: non-orientable $\Rightarrow \exists$ closed curve c, s.t.

free homotopy class of $c \rightarrow$ $[c]$ is nontrivial, and ~~parallel~~ parallel transportation ~~along~~ along each of it is orientation reversing.

compact $\Rightarrow$ $[c]$ $\exists$ minimizing closed geodesic γ.

$$\left.\begin{array}{l} n \text{ odd} \\ \boxed{\text{non orientable}} \ \det(P) = -1 \end{array}\right\} \overset{LA}{\Rightarrow} \left.\begin{array}{l} l \geq 2 \\ \sec > 0 \end{array}\right\} \overset{SVF}{\Rightarrow} \gamma \text{ can not be minimizing. } \lightning \ \square$$

Synge (b). (M^n, g) compact, $\sec > 0$, n even, nonorientable

$\Rightarrow \pi_1(M) = \mathbb{Z}_2$

Proof: M^n, nonorientable $\Rightarrow \exists \hat{M}^n$ is a double cover of M, orientable

(Lee) (M p T_pM O_p^1 O_p^2

$$\hat{M} = \{ (p, O_p) \mid p \in M, \ O_p \in \{O_p^1, O_p^2\}\}$$

$\pi : \hat{M} \to M$ Rie. covering map $\quad$?)

local isometry

$\hat{M}^n$ $\quad$ n even. compact, $\sec > 0$ orientable.

$\overset{\text{Synge(a)}}{\Longrightarrow}$ $\hat{M}^n$ is simply connected. $\Rightarrow \pi_1(M) = \mathbb{Z}_2$. $\square$

Remark: (1) existance of particular Rie. metric on a given smooth manifold.

(2) Examples. $\mathbb{R}P^n$ $\quad \pi : S^n \to \mathbb{R}P^n$

$\Rightarrow \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$

n even $\quad \mathbb{R}P^2$ compact $\sec > 0$, $\boxed{\text{non orientable}}$

$\boxed{\pi_1(\ ) = \mathbb{Z}_2}$

n odd $\quad \mathbb{R}P^3$ compact, $\sec > 0 \Rightarrow$ orientable.

(V). Space forms and Jacobi field.

· Space form: complete Rie mfld with constant sectional curvature.

Bonnet - Myers. $\sec \geq k > 0 \Rightarrow \text{diam} \leq \frac{\pi}{\sqrt{k}}$

$$\text{diam}\left(S^2\left(\frac{1}{\sqrt{k}}\right)\right) = \pi \cdot r = \pi \cdot \frac{1}{\sqrt{k}}$$

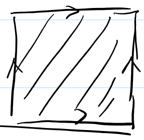
$$\sec(M^2) \geq \sec\left(S^2\left(\frac{1}{\sqrt{k}}\right)\right) \Rightarrow \text{diam}(M^2) \leq \text{diam}(S^2)$$

$$K_g \quad \boxed{K(Ag) = \frac{1}{A} K(g)}$$

$$\boxed{\text{Space form with sec. curvature} = 0, +1, -1.}$$

$\mathbb{R}^n$ $\quad \sec = 0$

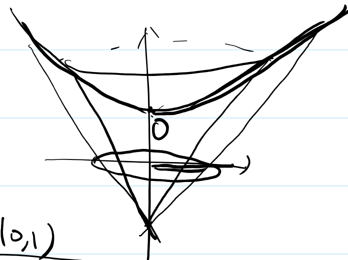
flat torus



Theorem 1. $\forall c \in \mathbb{R}$, $\forall n \in \mathbb{Z}^+$, $\exists !$ (up to isometries) simply connected n-dim'l space form with sectional curvature $= c$.

$c = 0, \quad +1, \quad -1$

$(\mathbb{R}^n, g_{\text{eucli}})$ $\quad (\mathbb{S}^n, g)$ $\quad (\mathbb{H}^n, g)$



$B^n(0,1)$

$$\boxed{\begin{aligned} R_{klij} = \frac{1}{2}\Big(&\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k}\Big) \\ &+ g_{mp}\left(\Gamma^m_{il}\Gamma^p_{jk} - \Gamma^m_{jl}\Gamma^p_{ik}\right) \end{aligned}}$$

Homework:

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§1. Geodesics in $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$.

$$\lim_{s \to \infty} \frac{e^s - 1}{e^s + 1} = 1$$

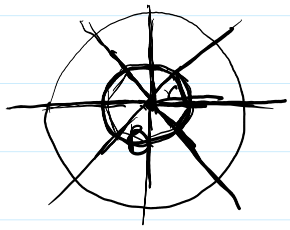
§1. Geodesics in $\mathbb{R}^n$, $\mathbb{S}^{n-1}$, $\mathbb{H}^{n}$.

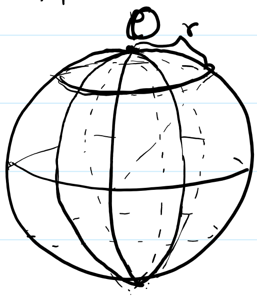
$\lim_{s\to\infty} \frac{e^s-1}{e^s+1}=1$

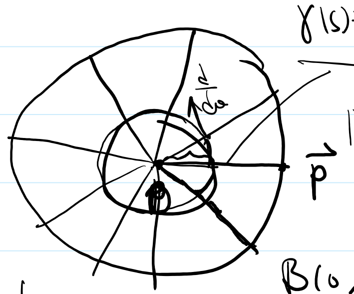
$\gamma(s)=\left(\frac{e^s-1}{e^s+1}\right)\vec{p}$

$|\gamma'(s)|_g=1$

$\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$







$\mathbb{R}^2$ $\qquad$ $\mathbb{S}^2$ $\qquad$ polar $\mathbb{H}^2$

diverge $\qquad$ diverge – converge $\qquad$ divergence

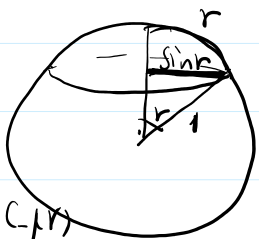
(M,g) $\quad C(r) := \{x \in M : d(x,0)=r\}$

Length denote $c_+(r)$, $c_-(r)$, $c_0(r)$ the length

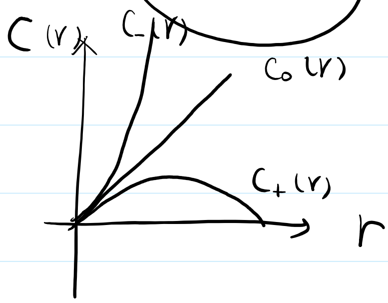
(1) $C_0(r) = 2\pi r$ $\quad$ linear in r.

(2) $C_+(r) = 2\pi \sin r$.

(3) $C_-(r) = 2\pi \sinh r$ $\quad$ exponential in r



curve. $\gamma(\theta) = (\rho_r, \theta)$, $\theta \in [0, 2\pi)$



$$C_-(r) = \int_0^{2\pi} \sqrt{\frac{4\rho_r^2}{(1-\rho_r^2)^2}}\,\Bigg|_{\rho_r = \frac{e^r-1}{e^r+1}} d\theta$$

$$g = \frac{4}{(1-\rho^2)^2}(d\rho^2 + \rho^2 d\theta^2)$$

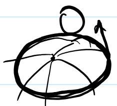
$$C_-(r) = \int_0^{2\pi} \frac{2\rho_r}{1-\rho_r^2}\,d\theta = 2\pi \cdot \frac{2\,\frac{e^r-1}{e^r+1}}{1-\left(\frac{e^r-1}{e^r+1}\right)^2} = 2\pi \underbrace{\boxed{\frac{e^r-e^{-r}}{2}}}_{\sinh r}$$

$$\cosh r = \frac{e^r+e^{-r}}{2}$$

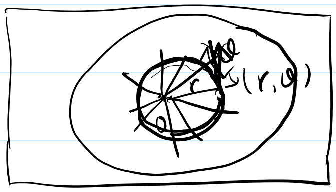
What happens in general?.

(M^n, g) $\quad 0 \in M$, $\quad$ geodesic sphere $\quad \exp_0 : T_0M \to M$

(M^n, g) $\quad O \in M$, $\quad$ geodesic sphere $\quad \exp_O : T_OM \to M$



$\xleftarrow{\exp}$



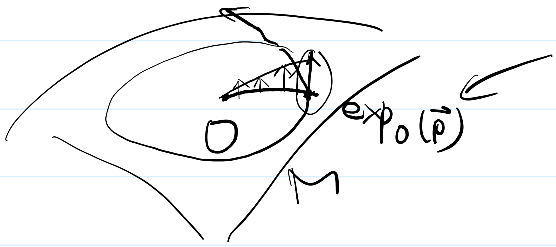
$d\exp_O(r,\theta) : T_{(r,\theta)}T_OM$

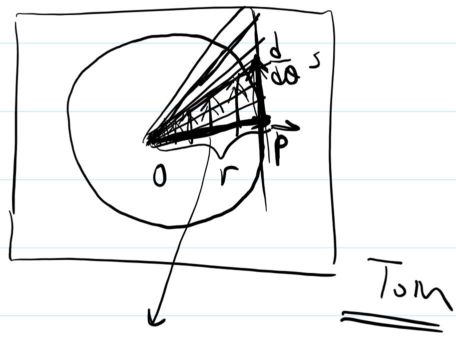
$T_OM \to T_{\exp_O(r,\theta)}M$

$c(r) = \text{length}\,\exp_O\big(\underbrace{(r,\theta)}_{r}\big)$

$$c(r) = \int_0^{2\pi} \left\langle d\exp_{O(r,\theta)}\left(\frac{d}{d\theta}\right), d\exp_{O(r,\theta)}\left(\frac{d}{d\theta}\right)\right\rangle^{1/2} d\theta$$

$\vec{p} = (r,\theta)$





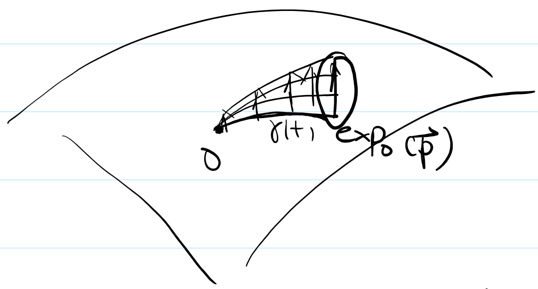
$t \longmapsto \frac{t}{r}\vec{p}$, $\quad t \in [0,r]$

$\gamma(t) = \exp_O\left(\frac{t}{r}\vec{p}\right)$, $\quad t \in [0,r]$, radial geodesic from O

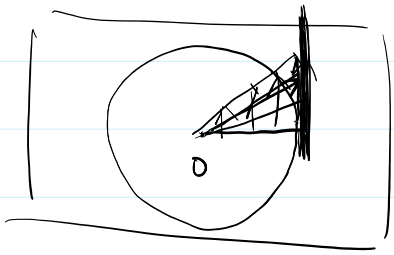
We consider: $\quad F : [0,r] \times (-\varepsilon, \varepsilon) \to M$

$$(t, s) \longmapsto \exp_O \frac{t}{r}\left(\vec{p} + s\frac{d}{d\theta}\right)$$

$F(t, 0) = \gamma(t)$



$\xleftarrow{\exp_O}$



$\gamma_s(t) = F(t,s)$ is a geodesic

variational field $\quad V(t) = \frac{\partial F}{\partial s}(t,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_O \frac{t}{r}\left(\vec{p} + s\frac{d}{d\theta}\right)$

? $\quad V(r) \overset{?}{=} d\exp_O(\vec{p})\left(\frac{d}{d\theta}\right)$

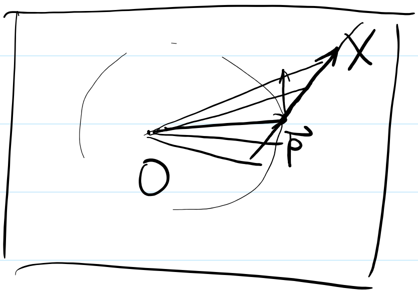
$$V(r) = \frac{\partial F}{\partial s}(r,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_O \frac{t}{r}\left(\vec{p} + s\frac{d}{d\theta}\right)\Bigg|_{t=r}$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \exp_O\left(\vec{p} + s\frac{d}{d\theta}\right)$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \exp_0 \underline{\left(\vec{p} + s \frac{d}{d\theta}\right)}$$

$$\boxed{\exp_0 : \underline{T_0M} \to \underline{M}} = d\exp_0(\vec{p})\left(\frac{d}{d\theta}\right) \qquad \square$$

We can be slightly more general.



$X \in T_{\vec{p}}(T_0M) \cong T_0M$.

$$F(t,s) = \exp_0 \frac{t}{r}(\vec{p} + sX)$$

$t \in [0,r]$, $s \in (-\varepsilon, \varepsilon)$

$$V(t) = \frac{\partial F}{\partial s}(t,0) \quad \text{variational field.}$$

Aim. $\underline{|V(r)|}$?

$$\frac{\partial F}{\partial t}(t,0) = \underline{T(t)}$$

$$\frac{D}{\partial t}\frac{D}{\partial t} V(t) = \frac{D}{\partial t}\underline{\frac{D}{\partial t}\frac{\partial F}{\partial s}}\cancel{(t,0)} = \underbrace{\frac{D}{\partial t}\frac{D}{\partial s}}\frac{\partial F}{\partial t}$$

$$= \frac{D}{\partial s}\underbrace{\boxed{\frac{D}{\partial t}\frac{\partial F}{\partial t}}}_{=0} + R\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right)\frac{\partial F}{\partial t}\Bigg|_{(t,0)}$$

$\frac{\partial F}{\partial t}(t,s) \equiv 0$ ~~tang~~ velocity field of $\gamma_s(\cdot)$

$s \in (-\varepsilon, \varepsilon)$

$$= R\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right)\frac{\partial F}{\partial t}\,(t,0)$$

$$\boxed{\nabla_T\nabla_T V = R(T,V)T} \quad \text{i.e.} \quad \boxed{\nabla_T\nabla_T V + R(V,T)T = 0}$$

<u>Definition</u> (Jacobi field) Let $\gamma : [a,b] \to M$ be a normal geodesic

~~Take~~ $T(t) := \dot\gamma(t)$

If a vector field $V(t)$ along γ satisfies

$$\nabla_T\nabla_T V + R(V,T)T = 0 \qquad (1)$$

then we call V a <u>Jacobi field</u> along γ.

then we call V a <u>Jacobi field</u> along γ.
The equation (1) is called the Jacobi equation.

How to solve the equation?

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