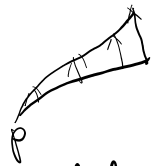
Jacobi field. $\gamma : [a,b] \to M$ be a geodesic

T : velocity field. $T(t) = \dot\gamma(t)$

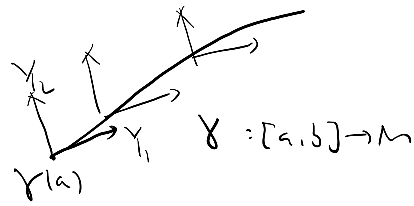
V is a C^∞ vector field along γ is called a Jacobi field if

$$\boxed{\nabla_T \nabla_T V + R(V,T)T = 0} \qquad (J)$$

Jacobi equation



parallel transport:



At $\gamma(a)$, pick $Y_1, \dots, Y_n \in T_{\gamma(a)}M$ orthonormal

parallel transport: $Y_1(t), \dots, Y_n(t)$

$$\boxed{V(t) = \sum_{i=1}^n f^i(t) Y_i(t)} \qquad \nabla_T V = \frac{df^i}{dt} Y_i(t) + f^i \underbrace{\nabla_T Y_i}_{=0}$$

$V(t)$

$$0 = \nabla_T \nabla_T V + R(V,T)T$$

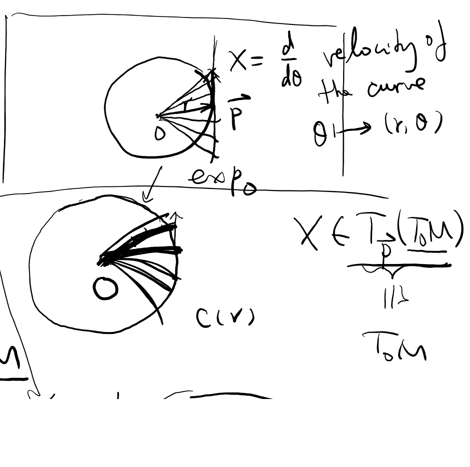
$$= \frac{d^2 f^i}{dt^2} Y_i(t) + f^i(t) R(Y_i(t), T)T$$

$$= \frac{d^2 f^j}{dt^2} Y_j(t) + \sum_j f^i \langle R(Y_i, T)T, Y_j\rangle Y_j$$

$$= \sum_j \left(\frac{d^2 f^j}{dt^2} + f^i \langle R(Y_i,T)T, Y_j\rangle \right) Y_j$$

$$\Rightarrow \quad (*) \begin{cases} \dfrac{d^2 f^j}{dt^2} + \sum_{i=1}^n \langle R(Y_i,T)T, Y_j\rangle f^i = 0, & j = 1, \dots, n \quad 2^{nd} \text{ linear ODE} \\ f^i(0) = 0 & i = 1, \dots, n \\ \dfrac{df^i}{dt}(0) = \langle \dfrac{X}{r}, Y_i(0)\rangle & i = 1, \dots, n \end{cases}$$

$[a,b] = [0,r]$



$$V(t) = f^i(t) Y_i(t) \qquad \boxed{V(0) = f^i(0) Y_i(0)}$$

$$\frac{DV}{dt}(0) = \sum_i \boxed{\frac{df^i}{dt}(0)} Y_i(0) = \sum_i \langle \frac{X}{r}, Y_i(0)\rangle Y_i(0)$$

$X \in T_0(T_0M) \cong T_0M$

$X \in T_0(T_0M) \cong T_0M$ $\quad c(r) \quad T_0M$

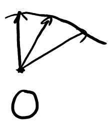
Recall from last time: $F(t,s) = \exp_0 \frac{t}{r}(\vec{p} + sX)$

$$V(t) = \frac{\partial F}{\partial s}(t,0), \quad V(r) = d\exp_0(\vec{p})(X)$$

$$V(0) = \frac{\partial F}{\partial s}(t,0)\Big|_{t=0} = \frac{\partial}{\partial s} F(t,s)\Big|_{s=0}\Big|_{t=0} = \frac{\partial}{\partial s} F(0,s)\Big|_{s=0} = 0$$

$$\frac{DV}{dt}(0) = \frac{D}{dt}\frac{\partial F}{\partial s}(t,s)\Big|_{t=0,s=0} \qquad t \mapsto \overset{\gamma(t)}{\exp_p tV} \quad \dot{\gamma}(0) = V$$

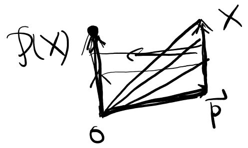
$$\overset{\text{torsion free}}{=} \frac{D}{\partial s}\underbrace{\frac{\partial F}{\partial t}(t,s)}\Big|_{t=0,s=0}$$



$V(t) = v^i(t)\frac{\partial}{\partial x^i}$ $\quad \frac{DV}{dt} = \frac{dv^i}{dt}(t)\frac{\partial}{\partial x^i}$

$$= \frac{D}{\partial s}\frac{1}{r}(\vec{p} + sX)\Big|_{s=0}$$

covariant derivative along the curve $s \mapsto F(0,s) \equiv 0$



$X \in T_{\vec{p}}(T_0M) \cong T_0(T_0M)$

$\vec{p} + sX \quad \vec{p} \in T_0(T_0M)$

$$= \frac{X}{r}$$

休息 10:38.

$\gamma : [0, r] \to M, \ \gamma(0) = 0$

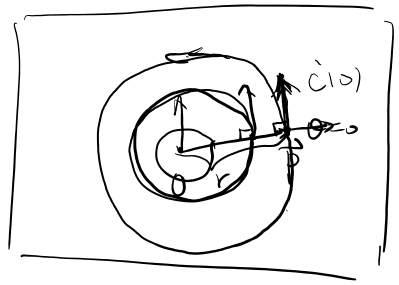
$F(t,s) := \exp_0 \frac{t}{r}(\vec{p} + sX) \qquad \exp_0 \frac{t}{r}\vec{p} = \gamma(t)$

$V(t) = \frac{\partial F}{\partial s}(t,0)$ variational field $\quad V(r)$

$V(t) = f^i(t) Y_i(t)$

$$\begin{cases} \dfrac{d^2 f^j}{dt^2} + \sum_{i=1}^n \langle R(Y_i, T)T, Y_j\rangle f^i(t) = 0, \quad j = 1, \dots, n \\ f^i(0) = 0 \\ \dfrac{df^i(0)}{dt} = \dfrac{1}{r}\langle X, Y_i(0)\rangle \end{cases}$$

$$X = \frac{d}{d\theta}\, c(\theta)\Big|_{\theta=0}$$



$c(\theta) = (r, \theta) \quad \theta \in [0, 2\pi]$

polar

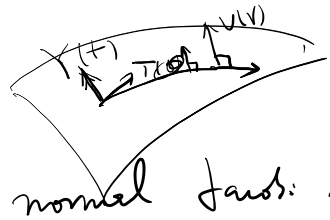
$$\frac{1}{r}\langle \dot c(0), \underline{Y_i(0)}\rangle$$

$C(\theta) = (r, \theta)$, $\theta \in [0, 2\pi]$

polar

$$\boxed{\mathbb{R}^2, \ \mathbb{S}^2, \ \mathbb{H}^2}$$

$$V(r) = d\exp_0(\vec p)(\dot c(0))$$



2-dim. Gauss lemma

~~$V(t) =$~~

$\langle V(t), T(t)\rangle = 0, \ \forall t$ normal Jacobi field

$\Rightarrow \boxed{V(t) = f(t)\,Y(t)}$ $\{Y, T\}$ orthonormal frame field along γ.

$$\begin{cases} \dfrac{d^2f}{dt^2} + \underbrace{\langle R(Y,T)T, Y\rangle}_{K} f(t) = 0 \\ f(0) = 0 \\ \dfrac{df}{dt}(0) = \dfrac{1}{r}\langle \underbrace{\dot c(0)}_{\| \atop r}, \boxed{Y(0)}\rangle = 1 \end{cases}$$

$$\dot c(0) \in T_{\vec p}(T_0M) \cong T_0(T_0M) \cong T_0M$$

$c(\theta) = (r, \theta)$ $|\dot c(0)| = r$

$= (r\cos\theta, r\sin\theta)$

On $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$

$K = 0, \ +1, \ -1.$

$$\boxed{\begin{cases} \dfrac{d^2f}{dt^2}(t) + K f(t) = 0 \\ f(0) = 0, \ \dot f(0) = 1 \end{cases}}$$

$$f(t) = \begin{cases} t & K = 0 \\ \sin t & K = 1 \\ \sinh t & K = -1 \end{cases}$$

$$C(r) = \int_0^{2\pi} \underbrace{\langle d\exp_0(\vec p)\left(\tfrac{d}{d\theta}\right), d\exp_0(\vec p)\left(\tfrac{d}{d\theta}\right)\rangle}_{\| \atop f^{\theta}(r)Y(r)}^{1/2} d\theta$$

$$= \int_0^{2\pi} \sqrt{f(r)^2}\, d\theta = 2\pi |f(r)| = 2\pi f(r)$$

$$= \begin{cases} 2\pi t & K = 0 \\ 2\pi \sin t & K = 1 \\ 2\pi \sinh t & K = -1 \end{cases}$$

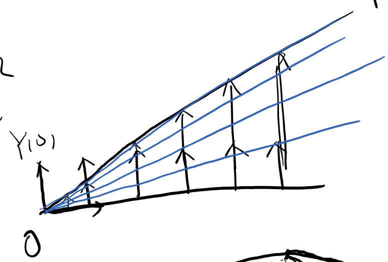
$\mathbb{R}^2$, $\mathbb{H}^2$, $\mathbb{S}^2$

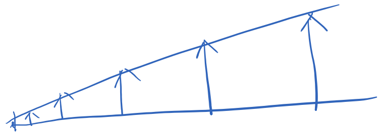
$f(t)Y(t) = \begin{cases} tY(t) & K = 0 \\ \ldots \end{cases}$

$\parallel$, $''$, ψ

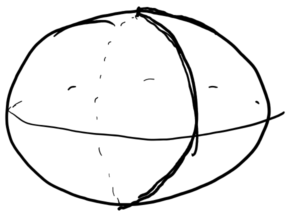
$$f(t)\,Y(t) = \begin{cases} t\,Y(t) & K=0 \\ \sin t\,Y(t) & K=+1 \\ \sinh t\,Y(t) & K=-1 \end{cases}$$

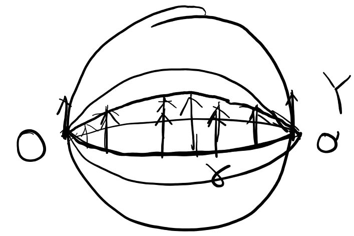
$\mathbb{R}^2$



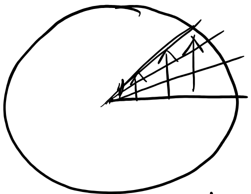


S^2





$\mathbb{H}^2$



$$\sinh t = \frac{e^t - e^{-t}}{2}$$

What is a Jacobi field?

$\gamma : [a,b] \to M^n$ geodesic

$$\frac{d^2 f^j}{dt^2} + f^i \langle R(Y_i, T)T, Y_j \rangle = 0, \quad j = 1, \dots, n$$

Prop. (1) Given $V, W \in T_{\gamma(a)}M$, $\exists!$ Jacobi field $U(t)$, $t \in [a,b]$ s.t.

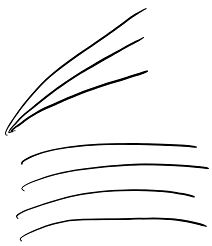
$$U(a) = V, \qquad \frac{DU}{dt}(a) = W.$$

(2). The linear space of all Jacobi fields along γ is of $2n$-dimensional.

Proof: Consequences of the ODE theory. $\square$

$F(t,s)$

$$\frac{D}{\partial t}\frac{\partial F}{\partial t} \equiv 0, \quad \forall s$$



Prop. Let $\gamma : [0,b] \to M$ be a geodesic, U be a vector field along γ.

U be a vector field along γ.

<u>Thm</u>. U is a Jacobi field <u>iff</u>. U is the variational field of a <u>geodesic variation</u>.

$$F(t,s) \quad \text{s.t.} \quad F(t,0) = \gamma(t)$$

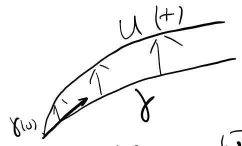
$$C^\infty \qquad F(t,s_0) = \gamma_{s_0}(t) \text{ is a geodesic } \forall s_0$$

<u>Proof</u>: $\Leftarrow$ <u>Excercise</u>.

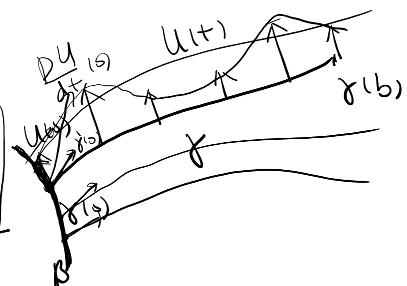
Compute $\nabla_T \nabla_T U$.

$\Rightarrow$ Let U be a Jacobi field along γ.

$U(0)=0$



$$\underline{F(t,s) = \exp_{\gamma(0)} t\left(\dot\gamma(0) + s\boxed{\frac{DU}{dt}(0)}\right)}$$



Let $\beta : (-\varepsilon, \varepsilon) \to M$ be a geodesic with

$\beta(0) = \gamma(0)$, $\dot\beta(0) = U(0)$, $\dot\gamma(0)$ $U(0)=0$ $\frac{DU}{dt}(0)$

$$\boxed{F(t,s) = \exp_{\beta(s)} t\left(\underbrace{V(s) + sW(s)}\right)}$$

where V, W are parallel vector field along β s.t.

$V(0) = \dot\gamma(0)$, $W(0) = \frac{DU}{dt}(0)$

<u>Remains</u>: $\boxed{\frac{\partial F}{\partial s}(t,0) = U(t)}$

$$\underbrace{Y(t)}_{\text{Jacobi field}} \qquad \underbrace{U(t)}_{\text{Jacobi field.}}$$

休息 11:28

$$Y(0) = \frac{\partial F}{\partial s}(t,0)\Big|_{t=0} = \frac{\partial F}{\partial s}(t,s)\Big|_{t=0,\, s=0}$$

$$= \frac{\partial F}{\partial s}(0,s)\Big|_{s=0} = \dot\beta(0)$$

$$= U(0)$$

$$\frac{DY}{dt}(0) = ?$$

$$= \frac{D}{\partial t}\left(\frac{\partial F}{\partial s}(t,0)\right)\Big|_{t=0} = \frac{D}{\partial t}\frac{\partial F}{\partial s}(t,s)\Big|_{s=0,\, t=0}$$

$\overset{\text{torsion free}}{=} \frac{D}{\partial s}\frac{\partial F}{\partial t}(t,s)\Big|_{s=0,\ t=0}$ $t(s)=s$

$= \frac{D}{\partial s}\underbrace{\left(V(s)+s\,W(s)\right)}\Big|_{s=0}$

a ~~vect~~ vector field along $\beta(s)$

$= 0 + W(s) + s\cdot \underbrace{\frac{DW}{\partial s}}_{=0}\Big|_{s=0}$

$= W(s)\Big|_{s=0} = W(0) = \frac{DU}{dt}(0)$ $\square$

SVF: variation $F(t,v,w)$, $t\in[a,b]$ $F(a,v,w)\equiv\gamma(a)$, $F(b,v,w)\equiv\gamma(b)$

"proper variation"

$$\underbrace{\frac{\partial^2}{\partial v\,\partial w}\Big|_{v=w=0}E(v,w)} = \int_a^b\left(\langle\nabla_T V,\nabla_T W\rangle - \underline{\langle R(W,T)T,V\rangle}\right)dt$$

$$= \boxed{I(V,W)}$$

symmetry $I(V,W)=I(W,V)$ Index form

bilinear

Prop: $\gamma:[a,b]\to M$ geodesic,

U is a Jacobi field along γ iff

$$\boxed{I(U,Y)=0,\ \forall \text{ vector field } Y \text{ along } \gamma \text{ with } Y(a)=Y(b)=0}$$

Proof: $I(U,Y) = \int_a^b\left(\langle\underbrace{\nabla_T U},\nabla_T Y\rangle - \underbrace{\langle R(U,T)T,Y\rangle}\right)dt$

$\overset{\nabla g=0}{=} \underbrace{\int_a^b \frac{d}{dt}\langle\nabla_T U,Y\rangle} - \langle\nabla_T\nabla_T U,Y\rangle - \langle R(U,T)T,Y\rangle\,dt$

$= \cancel{\langle\nabla_T U,Y\rangle\big|_a^b} - \int_a^b\langle\underbrace{\nabla_T\nabla_T U + R(U,T)T},Y\rangle\,dt$

U Jacobi $\Rightarrow I(U,Y)=0$

$I(U,Y)=0,\ \forall Y$, $Y(a)=Y(b)=0$ $\Rightarrow \nabla_T\nabla_T U + R(U,T)T=0$

Jacobi. $\square$

Prop: $\gamma:[a,b]\to M$ geodesic

Prop: $\gamma : [a,b] \to M$ geodesic

U is a Jacobi field iff U is a critical point of $\boxed{I(X,X)}$ ~~w.r.t. all~~.

That is, for any v.f. Y along γ with $Y(a)=Y(b)=0$

it holds $\boxed{\left.\frac{d}{ds}\right|_{s=0} I(X+sY, X+sY) = 0}$

Proof: $\left.\frac{d}{ds}\right|_{s=0} I(X+sY, X+sY)$

$= \left.\frac{d}{ds}\right|_{s=0} \Big(I(X,X) + sI(X,Y) + sI(Y,X) + s^2 I(Y,Y) \Big)$

$= I(X,Y) + I(Y,X)$

$= 2I(X,Y)$, $\forall Y$, $Y(a)=Y(b)=0$

$\square$

Prop. $\gamma : [a,b] \to M$ geodesic.

$T(t) := \dot\gamma(t)$

(1) The vector field $f(t)T(t)$ along γ is a Jacobi field iff f is linear

(2) Every Jacobi field U along γ can be written uniquely as

$U(t) = f(t)T(t) + U^{\perp}(t)$

where f is linear. $U^{\perp}$ is a Jacobi field s.t. $\langle U^{\perp}, T\rangle \equiv 0$

(3). If a Jacobi field U along γ satisfies

$\langle U(a), T(a)\rangle = 0$, $\langle U(b), T(b)\rangle = 0$

Then $\langle U, T\rangle \equiv 0$

Proof: $\nabla_T\nabla_T(fT) + R(fT, T)T = 0$

$= \frac{d^2 f}{dt^2} T + \underbrace{f R(T, T)T}_{=0}$

$$= \frac{d^2 f}{dt^2} T + \underbrace{f R(T, T) T}_{=0}$$

$$= \boxed{\frac{d^2 f}{dt^2} T}$$

fT Jacobi $\Longleftrightarrow$ $\frac{d^2 f}{dt^2} \equiv 0$ $\Longleftrightarrow$ f linear

(2) $U = fT + U^{\perp}$ unique

U Jacobi: $0 = \nabla_T \nabla_T U + R(U, T) T$

$$= \nabla_T \nabla_T (fT + U^{\perp}) + R(fT + U^{\perp}, T) T$$

$$(*) \quad 0 = \frac{d^2 f}{dt^2} T + \nabla_T \nabla_T U^{\perp} + \underbrace{f R(T, T) T}_{=0} + R(U^{\perp}, T) T$$

$$\langle U^{\perp}, T \rangle \equiv 0$$

$$0 \equiv \frac{d}{dt} \langle U^{\perp}, T \rangle \overset{\nabla g \equiv 0}{=} \langle \nabla_T U^{\perp}, T \rangle + \cancel{\langle U^{\perp}, \nabla_T T \rangle}^{\,0}$$

$$0 \equiv \frac{d}{dt} \langle \nabla_T U^{\perp}, T \rangle \overset{\nabla g \equiv 0}{=} \langle \nabla_T \nabla_T U^{\perp}, T \rangle$$

$$\Rightarrow \quad 0 = \frac{d^2 f}{dt^2} + \underbrace{\langle \nabla_T \nabla_T U^{\perp}, T \rangle}_{\equiv 0} + \underbrace{\langle R(U^{\perp}, T) T, T \rangle}_{=0}$$

$$\Rightarrow \quad \frac{d^2 f}{dt^2} \equiv 0 \quad \Rightarrow f \text{ linear !}$$

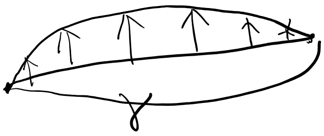
Back to $(*)$, we obtain $U^{\perp}$ is Jacobi □

(3). $U = fT + U^{\perp}$

$$\underbrace{\langle U(a), T(a) \rangle}_{= f(a)} = \underbrace{\langle U(b), T(b) \rangle}_{= f(b)} = 0$$

f linear

$\Rightarrow f \equiv 0 \quad \Rightarrow \quad U = U^{\perp}$. □



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