第二十四讲

2020年5月14日 13:29

Conjugate points

Jacobi fields. $\gamma: [a,b] \longrightarrow M^n$ geodesic

V is a vector field along γ s.t.

$$\nabla_T \nabla_T V + R(V,T)T = 0 \quad (\ast)$$

Jacobi field Then we call V a Jacobi field.

$\checkmark$ V is determined by $V(a)$, $\nabla_T V(a) \in T_{\gamma(a)}M^n$

dimension of the linear space of Jacobi field along γ $= 2n$

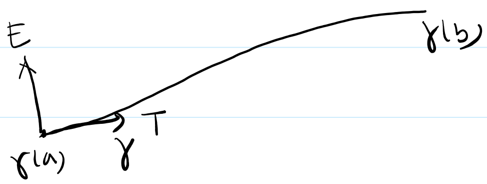
$$A: J_\gamma \longrightarrow T_{\gamma(a)}M \times T_{\gamma(a)}M \qquad \sum_{i=1}^n a^i U_i$$

$$U \longmapsto (U(a), \nabla_T U(a)) \qquad a^i \in \mathbb{R}$$

linear, injective (uniqueness), surjective (existence),

$\Rightarrow A$ is a linear isometry $\Rightarrow \dim J_\gamma = 2n$.

$\{E_1, E_2, \dots, E_n\} \subset T_{\gamma(a)}M$ orthonormal



Parallel transport.

$\{E_1(t), E_2(t), \dots, E_n(t)\} \leftarrow$ frame field along γ.

$\forall\ V$ C^∞ vector field along γ

$$V(t) = \sum_i f^i(t) E_i(t)$$

$\gamma: [a,b] \to M$ geodesic $\quad T(t) := \dot\gamma(t)$

$f(t)T(t)$ is Jacobian iff f linear.

$f(t) = at + b$, $a, b \in \mathbb{R}$

$(at+b)T(t)$ is a Jacobi field.

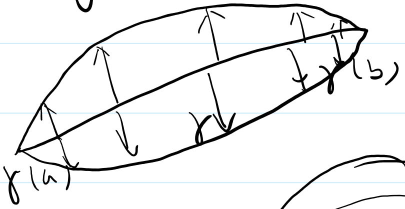
$\{(at+b)T(t),\ \forall a, b \in \mathbb{R}\} \leftarrow$

tT, T linearly indep. in the space

tT , T linearly indep. in the space

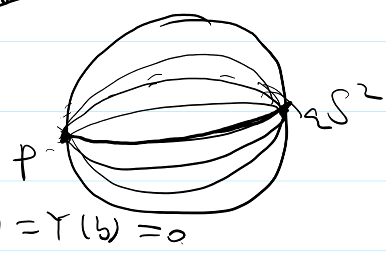
U Jacobi field with not identically zero

$\underline{U(a)=U(b)=0}$

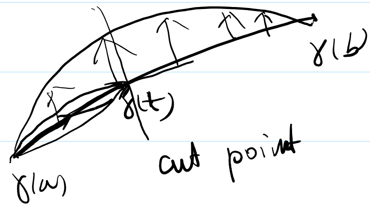


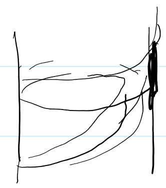
U is a Jacobian field

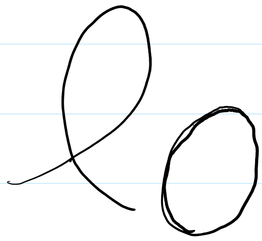
iff $I(U,Y)=0$, $\forall$ vector field along γ Y with $Y(a)=Y(b)=0$



$\boxed{I(U,U)=0}$







__Definition__ (conjugate point)

Let $\gamma:[a,b]\to M^n$ be a geodesic

For $t_0, t_1 \in [a,b]$, if $\exists$ a Jacobi field $U(t)$ along γ that does not vanish identically, but satisfies $\underline{U(a)=U(b)=0}$

Then t_0, t_1 are called __conjugate values__ along γ

We call $\gamma(t_0)$, $\gamma(t_1)$ are __conjugate points__ along γ.

The __multiplicity__ of t_0 and t_1 as conjugate values is defined as the dimension of the vector space

$$J'_\gamma=\{U \text{ a Jacobi field along } \gamma : U(a)=U(b)=0\}$$ □

__Remark__: $\dim J'_\gamma \leq n-1$

$\nabla_T U(a)$

$J'_\gamma \subset \boxed{\{U \text{ Jacobi: } \boxed{U(a)}=0\} \text{ vector space}}$ of dimension n

$\{\underbrace{c(t-a)}T, \ c\in\mathbb{R}\} \not\subset J'_\gamma$

~~$\{cT, \ c\in\mathbb{R}\}$~~

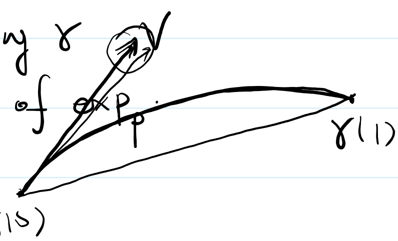
□

~~$\{c\,T,\ c\in\mathbb{R}\}$~~ $\square$

Thm. Let $\gamma:[0,1]\to M$ be a geodesic with $\gamma(0)=p\in M$.

$\dot\gamma(0)=V\in T_pM$. (i.e. $\gamma(t)=\exp_p tV$, $t\in[0,1]$)

Then 0 and 1 are conjugate values along γ if and only if $\boxed{V}$ is a critical point of $\exp_p$.



Rmk: $\exp_p: T_pM \to M$

If $V\in T_pM$, s.t. $d\exp_{p(V)}: T_V(T_pM)\to T_{\exp_p(V)}M$, $\exp_p(V)=\gamma(1)$

If $V\in T_pM$, s.t. $\exists\, 0\neq X\in T_V(T_pM)$,

$$d\exp_{p(V)}(X)=0$$

then we call V a critical point of $\exp_p$. $\square$

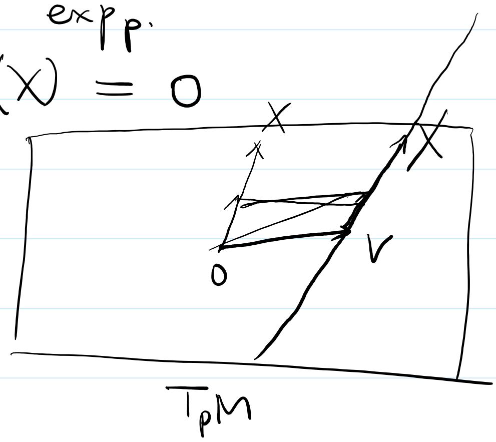
Proof. "$\Leftarrow$" Suppose V is a critical point of $\exp_p$.

$\exists\, 0\neq X\in T_V(T_pM)$, s.t. $d\exp_{p(V)}(X)=0$

$F:[0,1]\times(-\varepsilon,\varepsilon)\to M$

$$F(t,s):=\exp_p t\,\underbrace{(V+sX)}$$

(Let $c=c(s)$ be a curve in T_pM s.t. $c(0)=V$, $\dot c(0)=X$)



$$F(t,s):=\exp_p t\,c(s)$$

$U(t):=\frac{\partial F}{\partial s}(t,0)$ is a Jacobi field.

$$\underline{U(0)}=\frac{\partial F}{\partial s}(t,0)\Big|_{t=0}=\frac{\partial F}{\partial s}(t,s)\Big|_{t=0,s=0}=\frac{\partial F}{\partial s}(0,s)\Big|_{s=0}$$

$$=0$$

$$\underline{\nabla_T U(0)}=\frac{D}{\partial t}\left(\frac{\partial F}{\partial s}(t,0)\right)\Big|_{t=0}=\frac{D}{\partial t}\left(\frac{\partial F}{\partial s}(t,s)\right)\Big|_{t=0,s=0}$$

$$=\frac{D}{\partial s}\underbrace{\frac{\partial F}{\partial t}(t,s)}_{=c(s)}\Big|_{t=0,s=0}=\frac{D}{\partial s}c(s)\Big|_{s=0}=\dot c(0)=X$$

$\neq 0$

$\Rightarrow$ U is not identically zero. $\qquad \exp_p : \underbrace{T_pM}_{c(s)} \to M$

$$U(1) = \frac{\partial F}{\partial s}(1,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p t\, c(s) \Big|_{t=1}$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \exp_p c(s) = d(\exp_p)(c(0))(\dot{c}(0))$$

$$= d\exp_p(V)(X) = 0. \qquad \square$$

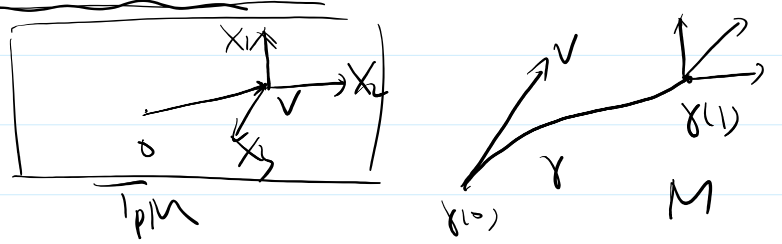
"$\Longrightarrow$" $\exists$ U Jacobi $U(0) = U(1) = 0$, $U \neq 0$ 休息 14:55

$\Rightarrow$ $\underset{\in T_pM}{V}$ is a critical point of $\exp_p$

Suppose V is not a critical point, i.e.

$$d\exp_p(V) : T_V(T_pM) \longrightarrow T_{\underset{\gamma(1)}{\exp_p(V)}} M$$

If $X_1, \dots, X_n \in T_V(T_pM)$ are linearly indep. vectors then $d\exp_p(V)(X_i)$, $i = 1, \dots, n$ $\in T_{\gamma(1)}M$ are also linearly independent.



$F_i(t,s) := \exp_p t(V + sX_i)$

$\hookrightarrow$ Jacobi field $U_i(t)$, $i = 1, \dots, n$ $\quad$ Recall $\begin{cases} U_i(0) = 0 \\ \nabla_T U_i(0) = X_i \end{cases}$

$U_1(t), \dots, U_n(t)$ linearly indep. Jacobi field, $U_i(0) = 0$

$$\dim\{U \text{ Jacobi along } \gamma \text{ with } U(0) = 0\} = n^{T_pM}$$

$\forall i \quad U_i(1) = d\exp_p(V)(X_i) \neq 0$

$\forall$ U Jacobi along γ with $U(0) = 0$, we have

$\forall\ U$ Jacobi along γ with $U(0)=0$, we have

$$U(t)=\sum_{i=1}^{n} a^i\, U_i(t)$$

$$U(1)=\sum_{i=1}^{n} a^i \underbrace{d\exp_p(v)(X_i)}_{\text{l. i.}} \neq 0. \quad ↯. \qquad \square$$

Index form $\gamma:[a,b]\to M$ geodesic

$$I(V,W)=\int_a^b\left(\langle\nabla_T V,\nabla_T W\rangle-\langle R(W,T)T,V\rangle\right)dt$$

$\mathcal{D}$: $V(t), W(t)$ C^∞ vector field along $\gamma(t)$

$\mathcal{V}$ $\quad \mathcal{D}\quad aV(t)+dW(t)$

$tV(t) \qquad V(t)$

$$I:\ \mathcal{V}\times\mathcal{V}\longrightarrow\mathbb{R}$$ symmetric bilinear

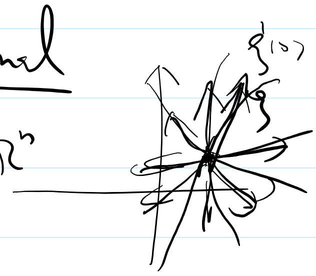
$\boxed{\text{Morse Index Theorem}}$

$\cdot\ \mathbb{R}^n \qquad f:\mathbb{R}^n\longrightarrow\mathbb{R}$ minimal

$p\in\mathbb{R}^n$

$\nabla f(p)=0$

Consider a curve ξ in $\mathbb{R}^n$, $\xi(0)=p\in\mathbb{R}^n$



$$\frac{d}{ds}\Big|_{s=0} f\circ(\xi(s))=0$$

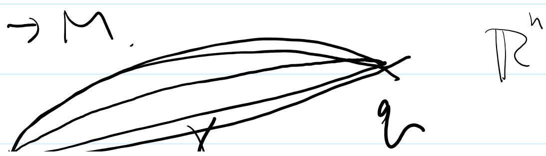
$$\frac{d^2}{ds^2}\Big|_{s=0} f(\xi(s)) = \mathrm{Hess} f(\dot\xi(0),\dot\xi(0))$$

$$\mathrm{Hess} f(v,v)>0,\ \forall\, v\in T_p\mathbb{R}^n,\ v\neq 0$$

$$\mathrm{Hess} f(v,w)=\frac{1}{2}\Big(\mathrm{Hess} f(v+w,v+w)-\mathrm{Hess} f(v,v) - \mathrm{Hess} f(w,w)\Big)$$

$$\mathrm{Hess} f: T_p\mathbb{R}^n\times T_p\mathbb{R}^n\longrightarrow\mathbb{R}^n$$

Space. $\mathcal{C}=\{$ all C^∞ curves $c:[a,b]\to M$, $c(a)=p,\ c(b)=q\}$



$c(a)=p,\ c(b)=q\}$

function $E:\mathcal{C}\to\mathbb{R}$

$\gamma_s(t)=F(t,s)$

E minimal at γ ?

$\{\gamma_s(t),\ s\in(-\varepsilon,\varepsilon)\}$ is a curve in $\mathcal{C}$

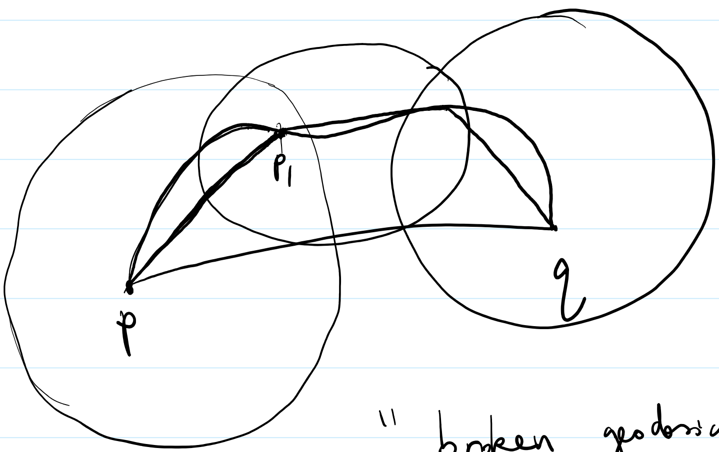
$E(s)=E(\gamma_s)$

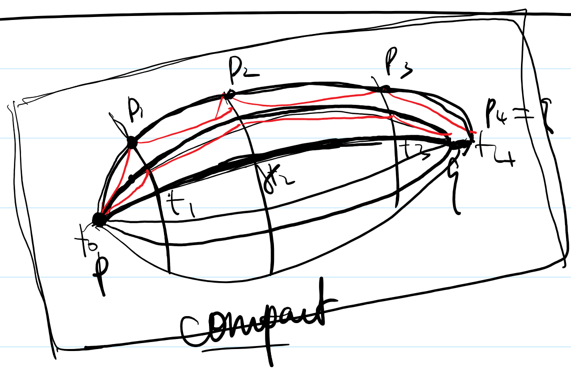
$E'(0)=0 \to$ geodesic

$E''(0) \cancel{0} = I(V,V) = \text{"Hess}E\text{"}(V,V)$

$\text{"Hess}E\text{"}(V,W)=\frac{1}{2}\{\text{"Hess}E\text{"}(V+W,V+W)-\text{"Hess}E\text{"}(V,V)-\text{"Hess}E\text{"}(W,W)\}$

$\text{"Hess}E\text{"}: \mathcal{V}\times\mathcal{V}\to\mathbb{R}$





"broken geodesics" $\gamma:[a,b]\to M$

$F:[a,b]\times(-\varepsilon,\varepsilon)\to M \longrightarrow \gamma_s:[a,b]\to M$

$a=t_0<t_1<t_2<\cdots<t_k<t_{k+1}=b$

$F(t,s)$ $\forall s$ γ_s p "broken geodesic"

$\mathcal{C}_{(p)\text{ broken geodesic}} \cong \underbrace{M^n\times M^n\times\cdots\times M^n}_{k}$

$\text{"Hess}E\text{"}: \mathcal{V}_i\times\mathcal{V}_i \longrightarrow \mathbb{R}$

"$\check{\text{Hess}}\,E$" : $V_{bg} \times V_{bg} \longrightarrow \mathbb{R}$

Index form finite dim'l vector space.

下课