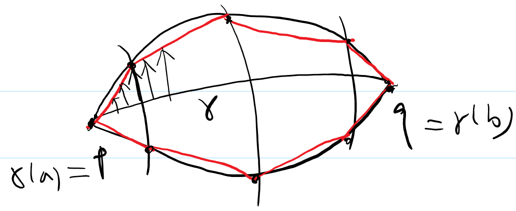
Index forms.

$F:[a,b]\times(-\varepsilon,\varepsilon)\to M$

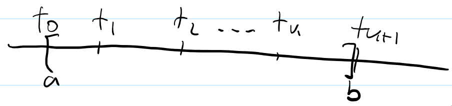
$(t,v)\mapsto F(t,v)$



$\gamma:[a,b]\to M$

$\gamma_v(t)=F(t,v)$ piecewise smooth

piecewise C^∞ variation.



SVF: $F:[a,b]\times(-\varepsilon,\varepsilon)\times(-\delta,\delta)\to M$

$(t,v,w)\longmapsto F(t,v,w)$

$\gamma_{v,w}(t)=F(t,v,w)$ piecewise smooth

$F(t,0,0)=\gamma(t)$ a geodesic

$V(t)=\frac{\partial F}{\partial v}(t,0,0)$

$W(t)=\frac{\partial F}{\partial w}(t,0,0)$

vector fields along $\gamma(t)$

$$\frac{\partial^2}{\partial v\partial w}\Big|_{(v,w)=(0,0)}E(v,w)=$$

$[a,b]$

$a=t_0<t_1<t_2<\cdots<t_k<t_{k+1}=b$

$\gamma_{v,w}\big|_{[t_i,t_{i+1}]}$ smooth, $i=0,\dots,k$

V,W piecewise smooth vector fields along γ.

$$\frac{\partial^2}{\partial v\partial w}E(v,w)\Big|_{(v,w)=(0,0)}=\frac{\partial^2}{\partial v\partial w}E(0,0)$$

$$=\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial w}\frac{1}{2}\int_a^b\left\langle\frac{\partial F}{\partial t},\frac{\partial F}{\partial t}\right\rangle dt\Big|_{(v,w)=(0,0)}$$

$$=\frac{\partial}{\partial v}\int_a^b\left\langle\frac{D}{\partial w}\frac{\partial F}{\partial t},\frac{\partial F}{\partial t}\right\rangle dt\Big|_{(v,w)=(0,0)}$$

$$\overset{\text{torsion free}}{=}\int_a^b\left\langle\frac{D}{\partial v}\frac{D}{\partial t}\frac{\partial F}{\partial w},\frac{\partial F}{\partial t}\right\rangle+\left\langle\frac{D}{\partial t}\frac{\partial F}{\partial w},\frac{D}{\partial t}\frac{\partial F}{\partial v}\right\rangle dt\Big|_{(v,w)=(0,0)}$$

$$=\int_a^b\underbrace{\left\langle\frac{D}{\partial t}\frac{D}{\partial v}\frac{\partial F}{\partial w},\frac{\partial F}{\partial t}\right\rangle}+\underbrace{\left\langle R\Big(\underset{V}{\frac{\partial F}{\partial v}},\underset{T}{\frac{\partial F}{\partial t}}\Big)\underset{W}{\frac{\partial F}{\partial w}},\underset{T}{\frac{\partial F}{\partial t}}\right\rangle}dt\Big|_{(v,w)=(0,0)}$$

$$+\int_a^b\underbrace{\langle\nabla_T W,\nabla_T V\rangle}dt$$

$$\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}$$

$\frac{D}{\partial t}\frac{\partial F}{\partial t}(t,0,0)=0$

$$=\int_a^b\frac{d}{dt}\left\langle\frac{D}{\partial v}\frac{\partial F}{\partial w},\frac{\partial F}{\partial t}\right\rangle dt\Big|_{(v,w)=(0,0)}+\int_a^b\langle\nabla_T W,\nabla_T V\rangle-\langle R(V,T)T,W\rangle dt$$

$$= \int_a^b \frac{d}{dt} \left\langle \frac{D}{\partial v} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle dt \Big|_{(v,w)=(0,0)} + \int_a^b \langle \nabla_T W, \nabla_T V \rangle - \langle R(V,T)\cancel{T}, W \rangle dt$$

$$= \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \underline{\frac{d}{dt} \langle \nabla_V W, T \rangle} \, dt + \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(V,T)T, W \rangle \right) dt$$

$$= \underline{\sum_{i=0}^{k} \left(\langle \nabla_V W, T \rangle |_{t_{i+1}} - \langle \nabla_V W, T \rangle |_{t_i} \right)} + I_a^b(V, W)$$

$\nabla_V W$: C^∞ v.f. T: C^∞ v.f.

$a = t_0, \; t_1, \; t_2, \; \dots, \; t_k, \; t_{k+1} = b$

$$= \langle \nabla_V W, T \rangle (b) - \langle \nabla_V W, T \rangle (a) + \underbrace{\sum_{j=1}^{k} \left(\langle \nabla_V W, T \rangle (t_j^-) - \langle \nabla_V W, T \rangle (t_j^+) \right)}_{=0}$$

$$+ I_a^b (V, W)$$

$$\Rightarrow \frac{\partial^2}{\partial v \partial w}\Big|_{(0,0)} E(v,w) = \langle \nabla_V W, T \rangle \Big|_a^b + I_a^b (V, W)$$

If $\exists$ a proper ~~var~~ piecewise C^∞ variation of a geodesic γ

Then $\frac{\partial^2}{\partial v \partial w}\Big|_{(0,0)} E(v,w) = I_a^b (V, W) = \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(V,T)T, W \rangle \right) dt$

Definition: The index form of a geodesic $\gamma : [a,b] \to M$ is a symmetric bilinear form s.t.

$$I(V,W) = \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(V,T)T, W \rangle \right) dt$$

for any piecewise smooth vector fields along γ.

Denote: $\boxed{\mathcal{V} := \{ \text{piecewise } C^\infty \text{ v.f. along } \gamma \}}$

real vector space

$$I : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$$
$$(V, W) \longmapsto I(V,W) \quad \text{symmetric bilinear}$$

~~$\forall\ \mathcal{V}$~~ $\{ I(V,V) : V \in \mathcal{V} \}$ quadratic form

LA: $\mathcal{V}^n$ n-dim'l vector space

$I : \mathcal{V}^n \times \mathcal{V}^n \to \mathbb{R}$ symmetric bilinear form

$\mathcal{V}^n \ \exists$ basis $(v_1, \dots, v_n)$

$\forall \ V \in \mathcal{V}^n, \ V = a^i v_i \quad (a^1, \dots, a^n)$

$\forall\ V \in \mathcal{V}^n,\quad V = a^i v_i \qquad (a^1, \dots, a^n)$

$$I(V,W) = (a^1 \dots a^n)\underbrace{\begin{pmatrix} +1 & & & & & & & \\ & \ddots & & & & & & \\ & & +1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \end{pmatrix}}_{\text{symmetric matrix}}\begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix}$$

p, q, $n-p-q$

null space

$$\{ V \in \mathcal{V}^n : I(V,W) = 0,\ \forall\ W \in \mathcal{V} \}$$

nullity $\quad n-p-q$

$\gamma : [a,b] \to M$ geodesic $\qquad p \quad q$

Notation: $\mathcal{V}_0 := \{ X \in \mathcal{V} : X(a) = X(b) = 0 \}$

Def (index and nullity of γ) $\quad \gamma : [a,b] \to M$ geodesic

null space $\quad \{ X \in \mathcal{V}_0 \mid I(X,Y) = 0,\ \forall\ Y \in \mathcal{V}_0 \}$

finite $\quad \leq n-1$

nullity $\quad N(\gamma) = \dim \{ X \in \mathcal{V}_0 \mid I(X,Y) = 0,\ \forall\ Y \in \mathcal{V}_0 \}$

$\boxed{N(\gamma) \leq n-1}$

finite $\to$ $\operatorname{Ind}(\gamma) = \max\dim \{ A \subset \mathcal{V}_0 \text{ subspace} \mid I \text{ is negatively definite on } A, \text{ i.e. } \forall\ X \in A,\ X \neq 0,\ I(X,X) < 0 \}$

index of γ

two $\to$ $\operatorname{Ind}_+(\gamma) = \max\dim \{ A \subset \mathcal{V}_0 \text{ subspace} \mid I \text{ is positively definite on } A \}$

$\gamma(a) \quad \gamma(b)$

$$\boxed{\begin{aligned} &\{ X \in \mathcal{V}_0 \mid I(X,Y) = 0,\ \forall\ Y \in \mathcal{V}_0 \} \\ \triangleq\ &\{ X \text{ Jacobian},\ X(a) = X(b) = 0 \} \end{aligned}} \Longrightarrow$$

$N(\gamma)$ is the multiplicity of a and b as conjugate values.

休息到 10:35. $\quad \leq n-1$

U is a Jacobi field along $\gamma : [a,b] \to M$ iff

$I(U,Y) = 0,\ \forall\ Y\ C^\infty$ v. f. along γ with $Y(a) = Y(b) = 0$

Prop: $\gamma : [a,b] \to M$ geodesic.

$U \in \mathcal{V}$ is a Jacobi field $\iff I(U,Y) = 0\ \forall$

$Y \in \mathcal{V}$ $\quad Y(a) = Y(b)$

$\underline{U \in \mathcal{V}}$ is a Jacobi field $\Leftrightarrow I(U,Y)=0 \ \forall \underbrace{Y \in \mathcal{V},\ Y(a)=Y(b)=0.}_{Y \in \mathcal{V}_0}$

$\underline{\text{Proof}}$: "$\Rightarrow$" U is a Jacobi field. Particularly, U C^∞

$$I(U,Y) = \int_a^b \langle \nabla_T U, \nabla_T Y\rangle - \langle R(U,T)T, Y\rangle \, dt$$

$$= \int_a^b \frac{d}{dt}\langle \nabla_T U, Y\rangle - \underbrace{\langle \nabla_T\nabla_T U, Y\rangle - \langle R(U,T)T, Y\rangle}_{=0} dt$$

$$= \int_a^b \frac{d}{dt}\langle \nabla_T U, Y\rangle \, dt$$

$$= \langle \nabla_T U, Y\rangle\Big|_a^b = 0, \ \forall Y \in \mathcal{V}_0.$$

"$\Leftarrow$" $U \in \mathcal{V}$, $I(U,Y)=0$, $\forall Y \in \mathcal{V}_0$.

$$I(U,Y) = \int_a^b \left(\langle \nabla_T U, \nabla_T Y\rangle - \langle R(U,T)T, Y\rangle\right) dt$$

$a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$. U, Y C^∞ on each (t_i, t_{i+1})

$$= \int_a^b \frac{d}{dt}\langle \nabla_T U, Y\rangle - \langle \nabla_T\nabla_T U + R(U,T)T, Y\rangle \, dt$$

$\lim_{t\to t_i^-} U(t) = \lim_{t\to t_i^+} U(t)$

$$= \sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\langle \nabla_T U, Y\rangle - \int_a^b \langle \nabla_T\nabla_T U + R(U,T)T, Y\rangle \, dt$$

$$\sum_{i=0}^{k} \langle \nabla_T U, Y\rangle\Big|_{t_i}^{t_{i+1}}$$

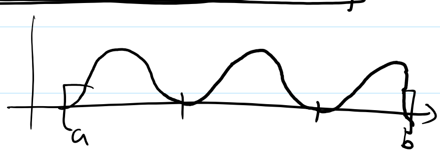
$$= \sum_{i=0}^{k}\left(\langle \nabla_T U(t_{i+1}^-), Y(t_{i+1})\rangle - \langle \nabla_T U(t_i^+), Y(t_i)\rangle\right)$$

$$= -\langle \nabla_T U(a^+), Y(a)\rangle + \langle \nabla_T U(b^-), Y(b)\rangle$$

$$+ \sum_{j=1}^{k}\langle \nabla_T U(t_j^-) - \nabla_T U(t_j^+), Y(t_j)\rangle$$

$$\boxed{\begin{aligned} I(U,Y) &= \langle \nabla_T U(b^-), \underbrace{Y(b)}_{=0}\rangle - \langle \nabla_T U(a^+), \underbrace{Y(a)}_{=0}\rangle \\ &+ \underbrace{\sum_{j=1}^{k}\langle \nabla_T U(t_j^-) - \nabla_T U(t_j^+), Y(t_j)\rangle} \\ &- \int_a^b \langle \underbrace{\nabla_T\nabla_T U + R(U,T)T}, Y\rangle \, dt \end{aligned}} \quad (\star)$$

Set $Y = f\left(\nabla_T\nabla_T U + \overset{0}{R}(U,T)T\right)$



Set $Y = f\,(\nabla_T\nabla_T U + R(U,T)T)$

$f : [a,b] \to \mathbb{R}$ C^∞ fct. $f(t_i) = 0$, $i = 0, \dots, k+1$.

otherwise $f > 0$.

$f(t_i) = 0$ $\nabla_T\nabla_T U(t_i)$? $\to$ $Y \in \mathcal{V}_0$

Insert Y into $(\star)$. $Y(t_j) = 0$

$$\Rightarrow I(U,Y) = -\int_a^b f\,|\nabla_T\nabla_T U + R(U,T)T|^2\,dt = 0$$

$f|_{(t_i,t_{i+1})}$ positive

$$\Rightarrow f\,|\nabla_T\nabla_T U + R(U,T)T|^2 = 0$$

$$\Rightarrow |\nabla_T\nabla_T U + R(U,T)T|^2 = 0 \text{ on each } (t_i, t_{i+1})$$

$\rightsquigarrow$ U is a piecewise Jacobi field.

$\rightsquigarrow$ $\forall Y \in \mathcal{V}_0$, $I(U,Y) = \sum_{j=1}^{k} \langle \nabla_T U(t_j^-) - \nabla_T U(t_j^+), Y(t_j) \rangle$

$\forall i_0 \in \{1, \dots, k\}$ Set $Y \in \mathcal{V}_0$, $\begin{cases} Y(t_j) = 0 \text{ for any } j \neq i_0 \\ Y(t_{i_0}) = \nabla_T U(t_{i_0}^-) - \nabla_T U(t_{i_0}^+) \end{cases}$

$$\Rightarrow 0 = I(U,Y) = |\nabla_T U(t_{i_0}^-) - \nabla_T U(t_{i_0}^+)|^2$$

$$\Rightarrow \nabla_T U(t_{i_0}^-) = \nabla_T U(t_{i_0}^+)$$

$\nabla_T U(t_i)$

$\rightsquigarrow$ U C^1 vector field along γ. $U(t_i)$

By the uniquess of the soln. of Jacobi eq.,
we know U ~~C^∞ vector~~ Jacobi field along γ. $\square$

$\forall X \in \mathcal{V}_0$, $X \neq 0$
$I(X,X) > 0$

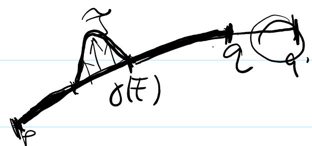
Thm. $\gamma : [a,b] \to M$, $p = \gamma(a)$, $q = \gamma(b)$

(1) $p = \gamma(a)$ has no conjugate point along γ $\Leftrightarrow$ $I : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathbb{R}$ positive definite.

(2) ~~$p=\gamma(a)$~~ $q = \gamma(b)$ is a conjugate point of $p = \gamma(a)$, $\forall t \in (a,b)$ $\gamma(a)$ and $\gamma(t)$ are not conjugate. (i.e. q is the first conjugate point of p)

$\gamma(a)$ and $\gamma(t)$ are not conjugate. (i.e. q is the first conjugate point of p)

$\Leftrightarrow$ I positive semidefinite but not positive definite



(3) $\exists\ \bar{t} \in (a,b)$ s.t. $p=\gamma(a)$ and $\gamma(\bar{t})$ are conjugate points $\Leftrightarrow$ $I(X,X)<0$ for some $X \in \mathcal{V}_0$.

~~ind$(\gamma)>0$~~

Rmk: If $\gamma(a)$ has no conjugate along $\gamma|_{[a,b]}$

then $\forall\ [\alpha,\beta] \subset [a,b]$, $\gamma(\alpha)$ has no conjugate pt along $\gamma|_{[\alpha,\beta]}$

Otherwise. $\exists\ \tilde{J}$ a nonzero Jacobi field along $\gamma|_{[\alpha,\beta]}$, s.t.

$\tilde{J}(\alpha)=\tilde{J}(\beta)=0$

$0 \neq J \in \mathcal{V}_0$ $\quad J|_{[a,\alpha]} \equiv 0$, $J|_{[\beta,b]} \equiv 0$, $J|_{[\alpha,\beta]}=\tilde{J}$.

$$0 < I_a^b(J,J) = I_\alpha^\beta(\tilde{J},\tilde{J}) = 0 \quad ⚡ \qquad \square$$

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Lemma: $\gamma:[a,b]\to M^n$ geodesic, $\gamma(a)$ has $\boxed{\text{no conjugate point along } \gamma.}$

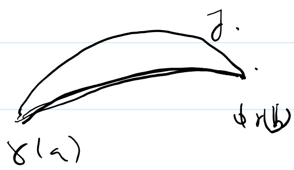
Then $\forall\ V_a \in T_{\gamma(a)}M$, $V_b \in T_{\gamma(b)}M$,

$\exists !$ Jacobi field U s.t.

$U(a)=V_a$, $U(b)=V_b$.

Proof. Given $U(a)=V_a$

$\mathcal{J}'=\{U \text{ Jacobi field},\ U(a)=V_a\}$



$\dim \mathcal{J}=n$

$\dim T_{\gamma(b)}M=n$

$$A: \mathcal{J}' \longrightarrow T_{\gamma(b)}M$$
$$U \longmapsto U(b)$$

A linear. $\boxed{\text{injective}}$. ✓

If A not injective, then $\exists\ U_1, U_2 \in \mathcal{J}'$. $U_1 \neq U_2$ $U_1(b)$

(If A not injective, then $\exists\ U_1, U_2 \in \mathcal{J}', U_1 \neq U_2, U_1(b) = U_2(b)$
$U_1 - U_2$ is a Jacobi field, non zero
$(U_1 - U_2)(a) = V_a - V_a = 0 \quad (U_1 - U_2)(b) = 0$ 矛)

$\Rightarrow A$ linear isometry, ~~$\dim \mathcal{J}' = \dim$~~ $\square$
$\Rightarrow A$ surjective

Proof of Thm
(1) " $\Rightarrow$ "
(2) " $\Rightarrow$ "
(3) " $\Rightarrow$ "
" $\Leftarrow$ "
" $\Leftarrow$ "
" $\Leftarrow$ "

$I : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathbb{R}$.

(1) " $\Rightarrow$ " no conjugate pt $\Rightarrow I > 0$

$\forall\ \boxed{U \in \mathcal{V}_0}$ we have to prove $\boxed{I(U, U) > 0}$
$U \neq 0$

Remark: Enough to consider $I : \mathcal{V}_0^\perp \times \mathcal{V}_0^\perp \to \mathbb{R}$
where $\boxed{\mathcal{V}_0^\perp = \{ X \in \mathcal{V}_0,\ \langle X, T \rangle \equiv 0 \}}$

$\mathcal{V}_0 = \mathcal{V}_0^\perp \oplus \{ fT,\ f : [a,b] \to \mathbb{R} \text{ piecewise smooth},\ f(a) = f(b) = 0 \}$

Claim $I(fT, fT) \geq 0$, " $=$ " iff $f \equiv 0$.

$\boxed{I(fT, U) = 0 \quad , \forall\ U \in \mathcal{V}_0^\perp}$

$\oplus$ decomposition $\forall\ U \in \mathcal{V}_0, \quad U = \underbrace{\langle U, T \rangle T} + \underbrace{(U - \langle U, T \rangle T)}$

Pf: $I(fT, fT) = \int_a^b \langle \nabla_T(fT), \nabla_T(fT) \rangle - \underbrace{\langle R(fT, T)T, fT \rangle}_{=0} dt$
$= \int_a^b (f'(t))^2\, dt \geq 0$

" $=$ " $\Leftrightarrow f'(t) \equiv 0$ on each $(t_i, t_{i+1}) \Rightarrow f \equiv 0$.

$I(fT, U) = \int_a^b \langle \nabla_T(fT), \nabla_T U \rangle - \underbrace{\langle R(fT, T)T, U \rangle}_{=0} dt$

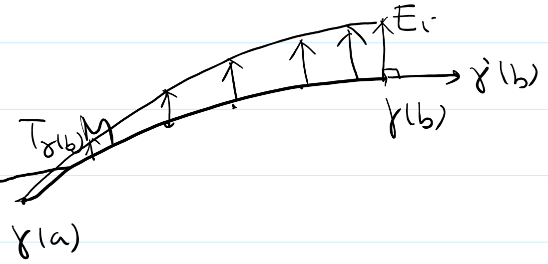
$= \int_a^b f'(t) \overbrace{\langle T, \nabla_T U \rangle}^{=0} dt \equiv 0$

$\langle U, T \rangle \equiv 0 \Rightarrow \frac{d}{dt} \langle U, T \rangle \equiv 0 \quad \langle \nabla_T U, T \rangle + \langle U, \underbrace{\nabla_T T}_{=0} \rangle = 0$

$\Rightarrow \langle \nabla_T U, T \rangle = 0$ $\square$

E_i

$$\Rightarrow \langle \nabla_T U, T\rangle = 0 \qquad \square$$



(1) " $\Rightarrow$ "

$\{\dot\gamma(b), E_2, \dots, E_n\}$ orthonormal in $T_{\gamma(b)}M$

normal Jacobi field J_i s.t.

$$J_i(a) = 0, \quad J_i(b) = E_i, \qquad i = 2, \dots, n$$

$$\langle J_i(a), T(a)\rangle = 0, \quad \langle J_i(b), T(b)\rangle = 0 \Rightarrow \langle J_i, T\rangle \equiv 0$$

$\mathcal{V}_0^\perp$ $T, J_2, \dots, J_n$

$\exists\, t_0 \in (a,b)$ s.t. $\sum_{i=2}^n a^i J_i(t_0) \neq 0$ $\Rightarrow$ $\sum_{i=2}^n a^i J_i \; (a) = 0, \; (t_0) = 0$

$\forall\, a^i \in \mathbb{R}$

$$\Rightarrow \sum_{i=2}^n a^i J_i \equiv 0$$

$$\sum_{i=2}^n a^i J_i(b) = \sum_{i=2}^n a^i E_i \neq 0 \quad \text{↯}$$

$\Rightarrow T, J_2, \dots, J_n$ frame field.

$\forall\, U \in \mathcal{V}_0^\perp$, $I(U,U) > 0$

$$U = \sum_{i=2}^n f^i(t) J_i(t), \quad f^i \text{ piecewise smooth}$$

$$I(U,U) = I\Big(\sum_{i=2}^n f^i J_i, \sum_{i=2}^n f^i J_i\Big)$$

$$= \int_a^b \langle \nabla_T (f^i J_i), \nabla_T (f^j J_j)\rangle - \langle R(f^i J_i, T)T, f^j J_j\rangle\, dt$$

$$= \int_a^b \langle \dot f^i J_i, \dot f^j J_j\rangle\, dt$$

$\nabla_T\nabla_T J_i = R(J_i, T)T$ $\forall i$

$$+ \int_a^b \langle \dot f^i J_i, f^j \nabla_T J_j\rangle + \int_a^b \langle f^i \nabla_T J_i, \dot f^j J_j\rangle$$

$$+ \int_a^b \langle f^i \nabla_T J_i, f^j \nabla_T J_j\rangle\, dt$$

$\nabla g \equiv 0$

Exercise. $0 =$

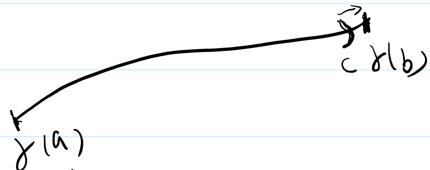
$$- \int_a^b f^i f^j \langle R(J_i, T)T, J_j\rangle\, dt$$

$$\frac{d}{dt}\left(f^i f^j \langle \nabla_T J_i, J_j\rangle\right)$$

$$I(U,U) = \int_a^b \langle \underline{\sum_{i=2}^n f^i J_i}, \sum_{j=2}^n f^j J_j \rangle \, dt \geq 0$$

"$=$" iff $f^i = 0$, $i = 2, \dots, n$; $f^i(a) = f^i(b) = 0$ $\Big\} \iff f^i \equiv 0 \iff U \equiv 0$. $\square$

(2) "$\Rightarrow$" $c \in (a,b)$



$\forall\, U \in \mathcal{V}_0^\perp$ Need: $\underline{I(U,U) \geq 0}$.

Not positive: $\exists$ nonzero Jacobi field U s.t. $U(a) = U(b) = 0$

$$I(U,U) = 0$$

frame field $\{\dot\gamma(t), E_2(t), \dots, E_n(t)\}$ parallel transp.

$$\forall\, U \in \mathcal{V}_0^\perp(a,b) \quad U = \sum_{i=2}^n f^i(t) E_i(t)$$

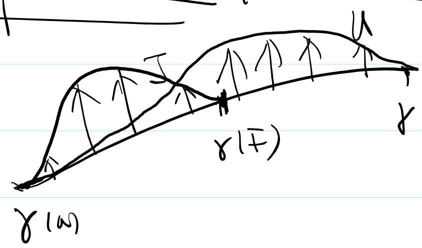
Define: $\tau: \mathcal{V}_0^\perp(a,b) \longrightarrow \mathcal{V}_0^\perp(a,c)$ $\quad s \in (a,c)$

$$U = \sum_{i=2}^n f^i(t) E_i(t) \longmapsto \underbrace{\sum_{i=2}^n f^i\Big(\underbrace{a + \frac{b-a}{c-a}(t-a)}_{=\, s}\Big) E_i\Big(a + \frac{b-a}{c-a}(t-a)\Big)}_{\tau(U)}$$

$$I_a^c(\tau(U), \tau(U)) \geq 0$$

$$\underline{I_a^b(U,U) = \lim_{c \to b} I_a^c(\tau(U), \tau(U))} \quad \boxed{\text{Exercise}}$$

(3) "$\Rightarrow$" $\exists$ nonzero Jacobi field J on $\gamma|_{[a,\bar t]}$ s.t.



$$J(a) = J(\bar t) = 0$$

$$\tilde J|_{[a,\bar t]} = J, \quad \tilde J|_{[\bar t, b]} = 0 \Rightarrow I(\tilde J, \tilde J) = I(J,J) = 0$$

Choose a $\underline{C^\infty\ U \in \mathcal{V}_0}$ $\quad U(a) = U(b)$

$$\mathcal{V}_0 \ni X = \frac{1}{c}\tilde J - cU, \quad c \text{ small number}$$

$$I(X,X) = I\Big(\frac{1}{c}\tilde J - cU, \frac{1}{c}\tilde J - cU\Big)$$

$$\overset{?}{<} 0 = \frac{1}{c^2}\underbrace{\cancel{I(\tilde J, \tilde J)}}_{=0} \boxed{-2I(\tilde J, U)} + c^2 \underbrace{I(U,U)}_{\text{independent of } c}$$

$I(\tilde J, U) = \int_a^b \dots$

$=0$ independent of c

$$I(\tilde{J}, u) = \int_a^b \langle \nabla_T \tilde{J}, \nabla_T u \rangle - \langle R(\tilde{J}, T)T, u \rangle \, dt$$

$$= \langle u(\bar{t}), \nabla_T \tilde{J}(\bar{t}^-) - \nabla_T \tilde{J}(\bar{t}^+) \rangle$$

$$u(\bar{t}) = \frac{\nabla_T \tilde{J}(\bar{t}^-) - \nabla_T \tilde{J}(\bar{t}^+)}{|\nabla_T \tilde{J}(\bar{t}^-) - \nabla_T \tilde{J}(\bar{t}^+)|^2} \Rightarrow I(\tilde{J}, u) = +1.$$

$$\Rightarrow I(X, X) = -2 + c^2 I(u, u) < 0. \qquad \square$$

下课