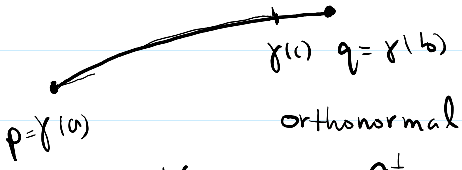


the first conjugate point of p

orthonormal frame field $\{\dot\gamma(t), E_2(t), \dots, E_n(t)\}$

$\forall\ U \in \mathcal{V}_0^\perp(a,b)\quad U=\sum_{i=2}^n f^i(t)E_i(t)$

Consider $\tau(U)\in\mathcal{V}_0^\perp(a,c)$ where $\boxed{\tau(U)=\sum_{i=2}^n \overbrace{f^i\big(\underbrace{a+\tfrac{b-a}{c-a}(t-a)}_{\in[a,b]}\big)}^{g^i(t)}E_i(t)}$ $\quad t\in[a,c]$

is a vector fields along $\gamma|_{[a,c]}$ with $\tau(U)(a)=0$

$\tau(U)(c)=0$.

$g^i(a)=f^i(a)=0$

$g^i(c)=f^i(b)=0$

Hence $I_a^c(\tau(U),\tau(U))>0$.

$\lim_{c\to b}\underbrace{I_a^c(\tau(U),\tau(U))}_{>0}=I_a^b(U,U)\geq 0$

$\gamma:[a,b]\to M$ normal geodesic $\Longrightarrow$ $I:\mathcal{V}_0\times\mathcal{V}_0\to\mathbb{R}$ ($\mathcal{V}_0^\perp$)

$\boxed{\text{no conjugate points of } \gamma(a) \iff I>0 \quad ①}$

$\gamma(b)$ the 1st conj. pt of $\gamma(a)$ $\iff$ $I\geq 0$, not positive

$\exists\ \bar t$ s.t. $\gamma(a)$, $\gamma(\bar t)$ are conj. $\iff$ $\exists X\in\mathcal{V}_0$, $X\neq 0$, $I(X,X)<0$

Lemma 2. $\gamma:[a,b]\to M$ geodesic, contains no conjugate pts.

Let U be a Jacobi field along γ, X be a piecewise C^∞ vector field along γ with $U(a)=X(a)$, $U(b)=X(b)$,

Then: $I(U,U)\leq I(X,X)$

where "=" holds iff $X=U$.

$\forall\ X\in\mathcal{V}_0(a,b)\ \ X\neq 0$

Check $I(X,X)>0$

$I(X,X)\geq I(U,U)=0 \quad U\in\mathcal{V}_0,\ U$ Jacobi

Remark: Basic Index Lemma

Proof: $X-U\in\mathcal{V}_0 \Rightarrow \begin{cases} I(X-U,X-U)\geq 0\\ \text{"=" holds iff } X-U\equiv 0\end{cases}$

$0\leq I(X-U,X-U)=I(X,X)-2I(X,U)+I(U,U)$

$I(\underset{\text{piecewise } C^\infty}{X},\underset{C^\infty}{U})=\int_a^b\langle\nabla_T X,\nabla_T U\rangle-\langle R(U,T)T,X\rangle\,dt$

$=\int_a^b \frac{d}{dt}\langle X,\nabla_T U\rangle\underbrace{-\langle X,\nabla_T\nabla_T U\rangle-\langle R(U,T)T,X\rangle}_{=0}dt$

$a=t_0<t_1<\cdots<t_k<t_{k+1}=b$

$=\int^b \frac{d}{dt}\langle X\ \nabla_T U\rangle\,dt$

$a=t_0<t_1<\cdots<t_k<t_{k+1}=b$ $\qquad = 0$

$$= \int_a^b \frac{d}{dt}\langle X, \nabla_T U\rangle \, dt$$

$$= \sum_{i=0}^{k} \langle X, \nabla_T U\rangle \Big|_{t_i}^{t_{i+1}} = \langle X, \nabla_T U\rangle(b) - \langle X, \nabla_T U\rangle(a)$$

$$= \langle U, \nabla_T U\rangle \Big|_a^b = I(U,U)$$

$$\Rightarrow \quad 0 \le I(X-U, X-U) = I(X,X) - I(U,U)$$

$$\Rightarrow \quad I(U,U) \le I(X,X)$$

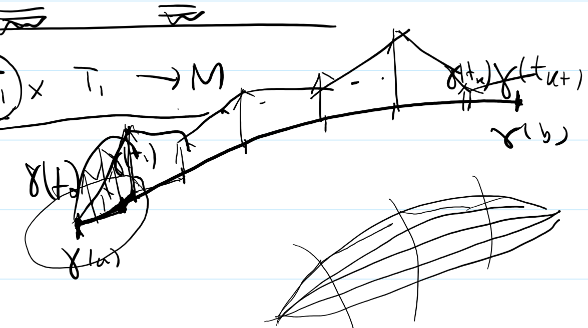
"=" holds iff $X-U \equiv 0$.

ind_+

$$\mathcal{V}_0 = \mathcal{V}_0^{\perp} \oplus \{fT, \ f(a)=f(b)=0\}$$

Fintiness of $\text{ind}(\gamma)$. $\qquad I: \mathcal{V}_0^{\perp} \times \mathcal{V}_0^{\perp} \longrightarrow M$

$\gamma: [a,b] \to M^n$ $\qquad I: T_1 \times T_1 \longrightarrow M$



Morse:

$\exists$ partition

$a=t_0<t_1<\cdots<t_k<t_{k+1}=b$

s.t. $\gamma|_{[t_i,t_{i+1}]}$ lies in a totally normal neighborhood,

Particularly $\gamma|_{[t_i,t_{i+1}]}$ contains no conjugate points

$\forall X \in \mathcal{V}_0^{\perp}$ $\qquad \dim = nk < \infty$

$$\varphi: \mathcal{V}_0^{\perp} \ T_1 \longrightarrow T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_k)}M$$

$$X_1 \longmapsto (X(t_1), \ldots, X(t_k))$$

X_2

linear map, 1-1 $\Rightarrow$ linear isometry $\Rightarrow$ $\dim T_1 = nk < \infty$

$$T_1 := \{X \in \mathcal{V}_0^{\perp} : X \text{ is Jacobian along } \gamma|_{[t_i,t_{i+1}]}, \ \forall i=0,\ldots,k\}$$

Claim: $\mathcal{V}_0^{\perp} = T_1 \oplus T_2$

$\forall X \in \mathcal{V}_0^{\perp}$, $\rightarrow$ $\underset{0}{X(t_0)}$, $X(t_1)$, $\ldots$, $X(t_k)$, $X(t_{k+1}) = 0$

$$J_X =: \varphi^{-1}(X(t_1), \ldots, X(t_k)) \in T_1$$

$$X = (J_X) + (X - J_X)$$

$$X = J_X + \underbrace{X - J_X}_{\in \mathcal{V}_0^\perp} \quad (X - J_X)(t_i) = 0, \quad i = 0, 1, \dots, k, k+1.$$

$T_2 := \{ X \in \mathcal{V}_0^\perp : X(t_i) = 0,\ i = 0, 1, \dots, k+1 \}$

$X \in T_1,\ X \in T_2 \Rightarrow X \equiv 0 \Rightarrow T_1 \cap T_2 = \{0\}$

$\Rightarrow \mathcal{V}_0^\perp = T_1 \oplus T_2.$ □

Claim. ① $I : T_2 \times T_2 \to \mathbb{R}$ $\quad I > 0$ ✓

② $\forall X_1 \in T_1,\ X_2 \in T_2,\quad I(X_1, X_2) = 0$ ✓

$$I(X_1, X_2) = \int_a^b \underbrace{\langle \nabla_T X_1, \nabla_T X_2 \rangle}_{\frac{d}{dt}\langle \nabla_T X_1, X_2\rangle - \langle \nabla_T \nabla_T X_1, X_2\rangle} - \langle R(X_1, T)T, X_2 \rangle \, dt$$

$$= \int_a^b \frac{d}{dt} \langle \nabla_T X_1, X_2 \rangle \, dt$$

$$= \sum_{i=0}^{k} \langle \nabla_T X_1, X_2 \rangle \Big|_{t_i}^{t_{i+1}} = 0 \quad ✓$$

$X_2(t_i) = 0,\ i = 0, \dots, k+1$

$\forall X \in T_2,\ X \neq 0$

$$I(X, X) = \sum_{i=0}^{k} \underbrace{\int_{t_i}^{t_{i+1}} \langle \nabla_T X, \nabla_T X \rangle - \langle R(X,T)T, X \rangle \, dt}_{I_{t_i}^{t_{i+1}}(X,X) \geq 0} \geq 0$$

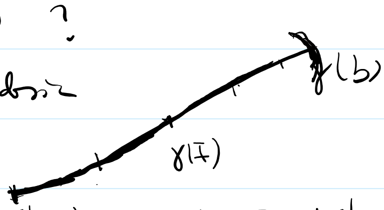
"=" holds iff $X \equiv 0$. □

$\Rightarrow \operatorname{Ind}(\gamma) \leq \dim(T_1) < \infty.$

休息 14:50.

$N(\gamma) \leq n-1 \qquad N(\gamma)$ = multiplicity of $\gamma(a)$ and $\gamma(b)$ as conjugate points.

$\operatorname{Ind}(\gamma) < \infty \qquad$ How large is $\operatorname{ind}(\gamma)$?

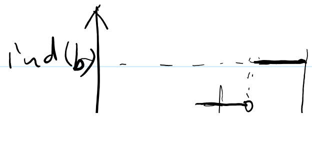


Morse Index Theorem. $\gamma : [a,b] \to M$ geodesic

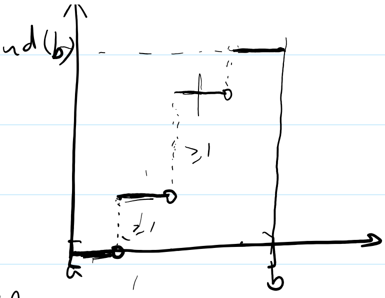
$$\boxed{\infty >} \operatorname{ind}(\gamma|_{[a,b]}) = \sum_{a < \bar{t} < b} \text{multiplicity of } a \text{ and } \bar{t} \text{ as conjugate values}$$

$\Leftrightarrow \operatorname{Ind}(\gamma|_{[a,b]}) \geq 1$

$$= \sum_{a < \bar{t} < b} N(\gamma|_{[a,\bar{t}]})$$



$$= \sum_{a<\bar t<b} N(\gamma|_{[a,\bar t]})$$



Rmk. index $\text{ind}(t) := \text{ind}(\gamma|_{[a,t]})$

$$\text{ind}: [a,b] \to \mathbb{R}_{\geq 0}$$

$$I: T_1 \times T_1 \longrightarrow \mathbb{R}$$

$\forall\, \varepsilon$ small

$$\begin{cases} \text{ind}(t-\varepsilon) = \text{ind}(t) \\ \text{ind}(t+\varepsilon) = \text{ind}(t) + N(\gamma|_{[a,t]}) \end{cases}$$

□

Cartan–Hadamard Thm.

$$I(V,V) = \int_a^b \langle \nabla_T V, \nabla_T V\rangle - \overbrace{\underbrace{\langle R(V,T)T, V\rangle}_{K(V,T)\,\leq 0}}^{\geq 0}\, dt > 0$$

$V \in U_0$

Thm. A complete, $\boxed{\text{simply-connected}}$ n-dim'l Rie. mfld (M,g) with $\boxed{\sec \leq 0}$ is diffeomorphic to $\mathbb{R}^n$. Moreover, $\boxed{\exp_p}: T_pM \to M$ is a diffeomorphism.

Rmk. Hadamard 1898 Cartan 1928

von Mangoldt 1881

Gromoll–Myers (1969) Any noncompact complete Rie mfld (M^n, g) with $\sec > 0$ is diffeomorphic to $\mathbb{R}^n$.

□

Thm. (M^n, g) complete Rie. mfld. Let $p \in M$ is a point s.t. no point of M is conjugate to p along any geodesic. Then $\exp_p: T_pM \to M$ is a covering map.

Proof: Remain to show $\exists\, \bar g$ s.t.

$$\left.\begin{array}{l} \exp_p: (T_pM, \bar g) \to (M,g) \text{ local isometry} \\ (T_pM, \bar g) \text{ complete.} \end{array}\right\}$$

M cpl. $\quad \exp_p: T_pM \to (M, g)$

$\bar g = \exp_p^* g$ $\qquad\qquad g$

$\forall X \in T_pM, \forall \ V \in T_X(T_pM)$, $\quad \bar g(V,V) = \exp_p^* g(V,V)$
$= g\big((d\exp_p)_{(X)}(V), (d\exp_p)_{(X)}(V)\big) \geq 0$

If $\bar g(V,V) = 0 \overset{?}{\Rightarrow} V = 0$

$\Updownarrow$

$(d\exp_p)_{(X)}(V) = 0 \Rightarrow V = 0$, $\quad \forall V \in T_X(T_pM)$

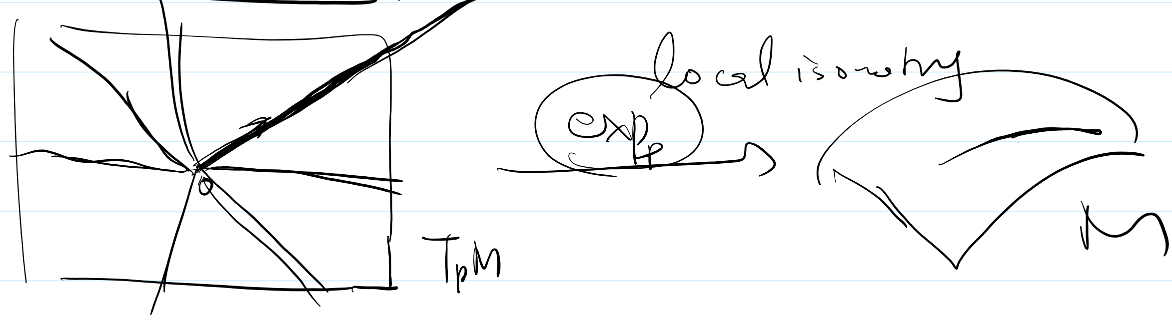
$(d\exp_p)_{(X)} : T_X(T_pM) \to T_{\exp_p X} M$

$V \longmapsto 0 \Rightarrow V = 0$

p has no conjugate pt. $\Rightarrow$

$\Rightarrow \bar g = \exp_p^* g$ is a Rie. metric on T_pM.

Remains to show: $(T_pM, \bar g)$ complete ✓

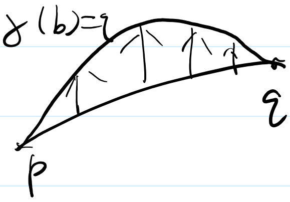


$\Rightarrow \exp_p : T_pM \to M$ is a covering map. □

Lemma: (M,g) Rie. mfld. sec ≤ 0

Then no two points are conjugate along any geodesic.

$\gamma : [a,b] \to M$ geodesic, $\gamma(a) = p$, $\gamma(b) = q$



Proof: Let U be a Jacobi field along γ s.t. $U(a) = U(b) = 0$

$\langle \nabla_T \nabla_T U, U \rangle + \underbrace{\langle R(U,T)T, U \rangle}_{K(U,T)} = 0$

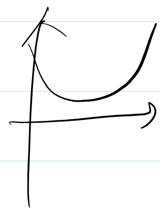
$$\langle \nabla_T \nabla_T U, U\rangle + \underbrace{\langle R(U,T)T, U\rangle}_{K(U,T)\ \le 0} = 0$$

$$\Rightarrow \boxed{\langle \nabla_T \nabla_T U, U\rangle = -K(U,T) \geq 0}$$

$$\frac{d^2}{dt^2}\langle U, U\rangle = \frac{d}{dt} 2\langle \nabla_T U, U\rangle$$

$$= 2\Big[\underbrace{\langle \nabla_T \nabla_T U, U\rangle}_{\geq 0} + \underbrace{\langle \nabla_T U, \nabla_T U\rangle}_{\geq 0}\Big] \geq 0.$$

U a C^∞ vector field along $\gamma : [a,b] \to M$



$$\langle U, U\rangle : [a,b] \to \mathbb{R}_{\geq 0}$$

$$t \mapsto \langle U(t), U(t)\rangle$$

$\frac{d^2}{dt^2}\langle U, U\rangle \geq 0 \Rightarrow \langle U, U\rangle(t)$ is a convex fcd.

$$\langle U, U\rangle(a) = \langle U, U\rangle(b) = 0$$

$$\langle U, U\rangle(t) \geq 0, \ \forall t$$

$$\Rightarrow \langle U, U\rangle(t) = 0, \ \forall t \in [a,b]$$

$\Rightarrow U \equiv 0.$ no conjugate $\square$

下课.