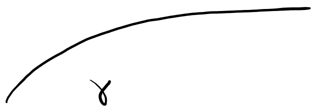
Comparison Theorem:

γ normal geodesic $\gamma = \gamma(t)$



normal Jacobi field $U = U(t)$

parallel frame field orthonormal $\{Y_1(t), \cdots, Y_n(t)\}$, $Y_1(t) = T(t) = \gamma'(t)$

$$U(t) = \sum_{i=2}^{n} \phi_i(t)\, \underline{\underline{Y_i(t)}}$$

$$\nabla_T \nabla_T U + R(U,T)T = 0$$

$$\Rightarrow \left\langle \sum_{i=2}^{n} \underline{\phi_i''(t)}\, Y_i(t) + \sum_{i=2}^{n} \phi_i(t)\, R(Y_i(t),T)T, Y_j(t) \right\rangle = 0, \quad j = 2, \cdots, n$$

$$\underbrace{\sum_{i=2}^{n} \phi_i''(t)\, \delta_{ij}}_{= \phi_j''(t)} + \sum_{i=2}^{n} \phi_i(t) \langle R(Y_i,T)T, Y_j \rangle = 0, \quad j = 2, \cdots, n$$

$$\phi_j''(t) + \sum_{i=2}^{n} \phi_i(t)\, \underline{\langle R(Y_i,T)T, Y_j \rangle} = 0, \quad j = 2, \cdots, n.$$

Constant sectional curvature c. $R(X,Y,X,Y) = c\, G(X,Y,X,Y)$

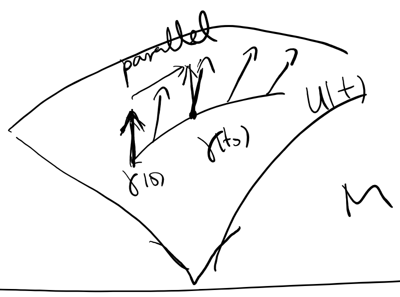
$\Rightarrow$ $R(X,Y,Z,W) = c\, G(X,Y,Z,W)$

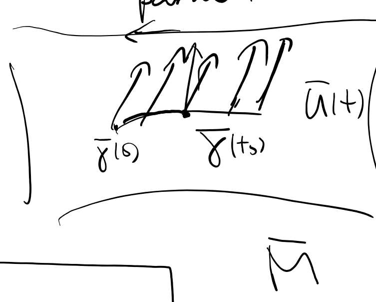
$$\phi_j''(t) + \sum_{i=2}^{n} c\, \phi_i(t) \underbrace{\left(\underline{\langle Y_i, T\rangle} \langle \underline{Y_j, T} \rangle - \underline{\langle T, T\rangle} \langle \underline{Y_i, Y_j} \rangle \right)}_{= \text{const.}} = 0$$

$j = 2, \cdots, n$

$$\frac{d^2}{dt^2}\left(\phi_2(t), \cdots, \phi_n(t)\right) + \left(\phi_2(t), \cdots, \phi_n(t)\right) \begin{pmatrix} \langle R(Y_i,T)T, Y_j \rangle \end{pmatrix}_{2 \le i,j \le n} = 0$$

$n=2$ Sturm–Liouville theory





(M^n, g), $(\bar{M}^n, \bar{g})$ two Rie mfld. ⊛

$\boxed{\begin{array}{l} (M, g), \quad (\bar M, \bar g) \quad \text{two Rie mfld.} \\ \gamma : [a,b] \to M \quad \bar\gamma : [a,b] \to \bar M \quad \underline{\text{normal}} \text{ geodesic.} \end{array}}$ $\circledast$

$$|U(t)|_g \geq |\bar U(t)|_{\bar g}$$

$\mathcal{V}_\gamma := \{\text{piecewise } C^\infty \text{ vector fields along } \gamma\}$, $\bar{\mathcal{V}}_{\bar\gamma}$

<u>Lemma</u>: $\Phi : \mathcal{V}_\gamma \to \bar{\mathcal{V}}_{\bar\gamma}$ a vector space isomorphism

s.t. $\forall X \in \mathcal{V}_\gamma$,

(1) $\langle X(t), \dot\gamma(t) \rangle_g = \langle \Phi(X)(t), \dot{\bar\gamma}(t) \rangle_{\bar g}$

(2) $\langle X(t), X(t) \rangle_g = \langle \Phi(X)(t), \Phi(X)(t) \rangle_{\bar g}$

(3) If $\nabla_T X$ is continuous at t, then $\nabla_{\bar T} \Phi(X)$ is continuous at t

$$\underline{\langle \nabla_T X, \nabla_T X \rangle_g = \langle \nabla_{\bar T} \bar\Phi(X), \nabla_{\bar T} \bar\Phi(X) \rangle_{\bar g}}$$

<u>Proof</u>: $\phi_{t_0} : T_{\gamma(t_0)} M \to T_{\bar\gamma(t_0)} \bar M$

Parallel transport.

Let $Y_1(t), \ldots, Y_n(t)$ parallel orthonormal frame field along γ
$Y_1(t) = \dot\gamma(t)$

Let $\bar Y_1(t), \ldots, \bar Y_n(t)$ " " along $\bar\gamma$
$\bar Y_1(t) = \dot{\bar\gamma}(t)$

Define $\phi_{t_0} : T_{\gamma(t_0)} M \to T_{\bar\gamma(t_0)} \bar M$

$$Y_i(t_0) \longmapsto \phi_{t_0}(Y_i(t_0)) = \bar Y_i(t_0)$$

$$\begin{array}{c} \downarrow \\ \phi_t : T_{\gamma(t)} M \longrightarrow T_{\bar\gamma(t)} \bar M \end{array}$$

$T_{\gamma(t)} M \xrightarrow{\text{parallel}} T_{\gamma(t_0)} M \xrightarrow{\phi_{t_0}} T_{\bar\gamma(t_0)} \bar M \xrightarrow{\text{parallel}} T_{\bar\gamma(t)} \bar M$

$$X \longmapsto \Phi(X), \quad \Phi(X)(t) := \phi_t(X(t))$$

$$\forall X \in \mathcal{V}_\gamma, \quad \underline{X(t)} = \boxed{\sum_{i=1}^n \underline{f^i(t)}\, \underline{Y_i(t)}}$$

$$\Phi(X)(t) = \phi_t(X(t)) = \boxed{\sum_{i=1}^n f^i(t)\, \underline{\bar Y_i(t)}}.$$

$$\Phi(X)(t) = \varphi_t(X(t)) = \left| \sum_{i=1}^{n} f^i(t)\, \bar{Y}_i(t) \right|$$

$$\langle \Phi(X), \dot{\bar{\gamma}}(t) \rangle_{\bar{g}} = f^1(t) = \langle X, \dot{\gamma}(t) \rangle_g$$

$$\langle \Phi(X), \Phi(X) \rangle = \sum_{i=1}^{n} f^i(t)^2 = \langle X, X \rangle_g$$

$$\nabla_T \Phi(X) = \sum_{i=1}^{n} \dot{f}^i(t)\, \bar{Y}_i(t)$$

$$\langle \nabla_T \Phi(X), \nabla_T \Phi(X) \rangle_{\bar{g}} = \sum_{i=1}^{n} \dot{f}^i(t)^2 = \langle \nabla_T X, \nabla_T X \rangle_g \quad \square$$

$V(t)$ variational field

$$I(V(t), V(t)) = \int_a^b \left\{ \underbrace{\langle \nabla_T V, \nabla_T V \rangle}_{\text{preserved}} - \langle R(V,T)T, V \rangle \right\} dt$$

$$\langle R(V,T)T, V\rangle = \boxed{K(V,T)}\left(\langle V,V\rangle\langle T,T\rangle - \langle V,T\rangle^2\right)$$

Thm ⊛

If $\forall\, t \in [a,b]$, $\forall\, \Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\forall\, \Pi_{\bar{\gamma}(t)} \subset T_{\bar{\gamma}(t)}\bar{M}$

we have $\quad K(\Pi_{\gamma(t)}) \leq \bar{K}(\Pi_{\bar{\gamma}(t)})$

Then $\Rightarrow \operatorname{ind}(\gamma) \leq \overline{\operatorname{ind}}(\bar{\gamma})$

In fact, If $I(W,W) < 0$ for some $W \in \mathcal{V}_0(a,b)$

then $\bar{I}(\Phi(W), \Phi(W)) < 0$, $\quad \Phi(W) \in \bar{\mathcal{V}}_0(a,b)$.

Prof: $\quad W \in \mathcal{V}_0(a,b) \xrightarrow{\Phi} \Phi(W) \in \bar{\mathcal{V}}_0(a,b)$

$$I(W,W) = \int_a^b \langle \nabla_T W, \nabla_T W \rangle - \underbrace{\langle R(W,T)T, W \rangle}_{} dt$$

$$\langle R(W,T)T, W \rangle = K(W,T)\left(\langle W,W\rangle\langle T,T\rangle - \langle W,T\rangle^2\right)$$

$$\bar{I}(\Phi(W), \Phi(W)) = \int_a^b \langle \nabla_T \Phi(W), \nabla_T \Phi(W) \rangle - \underbrace{\langle R(\Phi(W), \bar{T})\bar{T}, \Phi(W) \rangle}_{} dt$$

$$\langle R(\Phi(W), \bar{T})\bar{T}, \Phi(W) \rangle = K(\Phi(W), \bar{T})\left(\langle \Phi(W), \Phi(W)\rangle \langle \bar{T}, \bar{T}\rangle - \langle \Phi(W), \bar{T}\rangle^2\right)$$

$$K(W,T) \leq K(\Phi(W), \bar{T})$$

$$\Rightarrow 0 > I(W,W) \geq \bar{I}(\Phi(W), \Phi(W))$$

$\square$

Thm (Rauch Comparison Theorem)

⊛

(*)

Let $U, \bar{U}$ normal Jacobi field along $\gamma, \bar{\gamma}$ with

$$0 = U(a) = \bar{U}(a) = 0, \quad |\nabla_T U(a)|_g = |\nabla_T \bar{U}(a)|_{\bar{g}}$$

Suppose (1) If $\forall t \in [a,b]$, $\forall \Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\forall \Pi_{\bar{\gamma}(t)} \subset \bar{M}$

we have $$K(\Pi_{\gamma(t)}) \leq \bar{K}(\Pi_{\bar{\gamma}(t)})$$

(2) $\bar{\gamma}$ has no conjugate point on $[a,b]$

Then $$|U(t)|_g \geq |\bar{U}(t)|_{\bar{g}}, \quad \forall t \in [a,b]$$

Remark: $U, \bar{U}$ normal $\Leftarrow$ $\boxed{\langle \nabla_T U(a), \dot{\gamma}(a)\rangle = \langle \nabla_T \bar{U}(a), \dot{\gamma}(a)\rangle = 0}$

$$U = fT + U^{\perp}, \quad f \text{ linear} \quad U(a)=0 \Rightarrow f(a)=0$$

$f'(a) = 0 \qquad\qquad f \equiv 0$

Proof: If $\bar{U} \equiv 0$, trivial ✓

If $\bar{U} \not\equiv 0$, $\Rightarrow$ $\bar{U}(t) \neq 0$, $\forall t \in (a,b]$ ($\bar{\gamma}$ has no conj. pt.)

~~Remarks~~ Claim: ① $\lim\limits_{t\to a} \dfrac{|U(t)|_g^2}{|\bar{U}(t)|_{\bar{g}}^2} = 1$, ② $\dfrac{d}{dt} \dfrac{|U(t)|_g^2}{|\bar{U}(t)|_{\bar{g}}^2} \geq 0$, $\forall t \in (a,b]$

$\Rightarrow |U(t)|_g \geq |\bar{U}(t)|_{\bar{g}}$.

① $$\lim_{t\to a} \frac{\langle U(t), U(t)\rangle_g}{\langle \bar{U}(t), \bar{U}(t)\rangle_{\bar{g}}} = \lim_{t\to a} \frac{\cancel{2}\langle U(t), \nabla_T U(t)\rangle_g}{\cancel{2}\langle \bar{U}(t), \nabla_T \bar{U}(t)\rangle_{\bar{g}}}$$

$$= \lim_{t\to a} \frac{\langle \nabla_T U(t), \nabla_T U(t)\rangle + \langle U(t), \nabla_T\nabla_T U(t)\rangle}{\langle \nabla_T \bar{U}(t), \nabla_T \bar{U}(t)\rangle + \langle \bar{U}(t), \nabla_T\nabla_T \bar{U}(t)\rangle}$$

$$= 1.$$

② $$\frac{d}{dt} \frac{\langle U(t), U(t)\rangle_g}{\langle \bar{U}(t), \bar{U}(t)\rangle_{\bar{g}}} = \frac{2\langle U(t), \nabla_T U(t)\rangle \langle \bar{U}, \bar{U}\rangle - 2\langle U, U\rangle \langle \bar{U}, \nabla_T \bar{U}\rangle}{\langle \bar{U}(t), \bar{U}(t)\rangle^2} \geq 0$$

$\forall t \in (a,b]$

$\forall t_0 \in (a,b]$

$\forall t \in (a,b]$

$\forall t_0 \in (a,b]$

Remains: $\langle U, \nabla_T U\rangle(t_0)\,\langle \bar U,\bar U\rangle(t_0) \overset{?}{\geq} \langle U,U\rangle(t_0)\,\langle \bar U, \nabla_T \bar U\rangle(t_0)$

$$I_a^{t_0}(U,U) = \int_a^{t_0} \left(\langle \nabla_T U, \nabla_T U\rangle - \langle R(U,T)T,U\rangle\right) dt$$

$$= \int_a^{t_0} \frac{d}{dt}\langle U,\nabla_T U\rangle - \underbrace{\langle U, \nabla_T\nabla_T U\rangle - \langle R(U,T)T,U\rangle}_{0}\, dt$$

$$= \langle U,\nabla_T U\rangle \Big|_a^{t_0}$$

$$= \langle U,\nabla_T U\rangle(t_0)$$

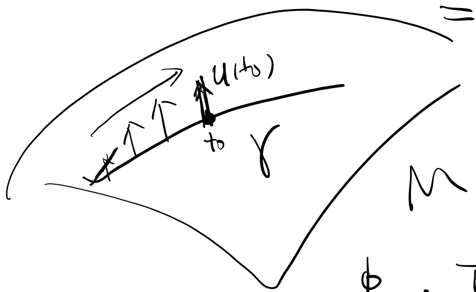
$$\boxed{\text{Remains.}\quad I_a^{t_0}(U,U) \geq \frac{|U(t_0)|^2}{|\bar U(t_0)|^2}\, \bar I_a^{t_0}(\bar U,\bar U)}$$

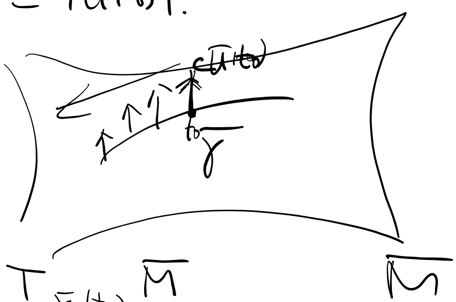
$\gamma|_{[a,t_0]}$ $\qquad c^2,\ c>0$

$$= \bar I_a^{t_0}(c\bar U, c\bar U)$$

$$|U(t_0)| = |c\bar U(t_0)|$$

$$= c\cdot|\bar U(t_0)| = |U(t_0)|.$$





$$\phi_{t_0}: T_{\gamma(t_0)}M \to T_{\bar\gamma(t_0)}\bar M$$

$$\phi_{t_0}(U(t_0)) = \underline{c\bar U(t_0)}$$

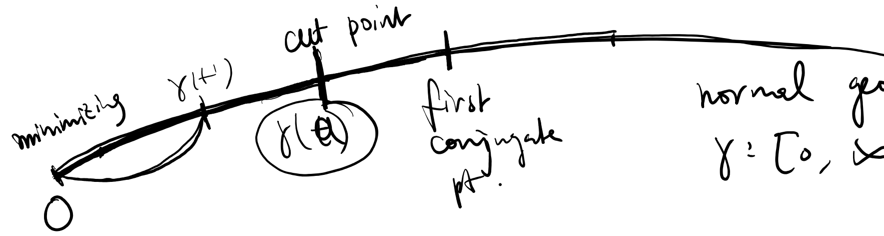
$$\rightsquigarrow \Phi(U)(t) := \phi_t(U(t))$$

$\Phi(U)$ is a vector field along $\bar\gamma$.

$$\boxed{I_a^{t_0}(U,U) \geq \bar I_a^{t_0}(\Phi(U),\Phi(U)) \underbrace{\geq}_{\text{minimizing property of Jacobi field.}} \bar I_a^{t_0}(c\bar U, c\bar U)}$$

$\square$

Cut point and Cut locus.



normal geodesic

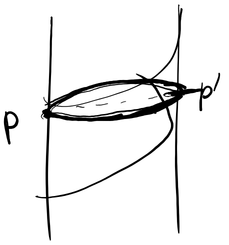
$\gamma: [0,\infty) \to M$

O

$$A = \{ t>0,\ \underline{d(O, \gamma(t)) = t} \}$$

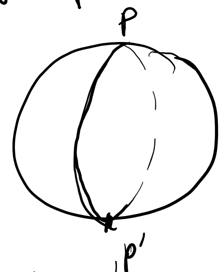
A either $(0, \infty)$ or $(0, a]$ for some $a>0$

<u>cut locus</u> of $O \in M$, $C(O) = \{ p \in M,\ p$ is a cut point of O along some geodesic γ from $O \}$



$\mathrm{Sec} = 0$

no <u>conjugate point</u>



<u>Thm</u>. (M, g) $\boxed{\text{complete,}}$ let $\gamma : [0, \infty) \to M$ be a <u>normal</u> geodesic with cut point $\gamma(a)$. Then, at least one of the following holds.

(1) a is the first conjugate value of 0 along γ.

(2) $\exists$ at least two minimizing geodesic from $p = \gamma(0)$ to $q = \gamma(a)$, and a is the minimal value s.t. this case happens

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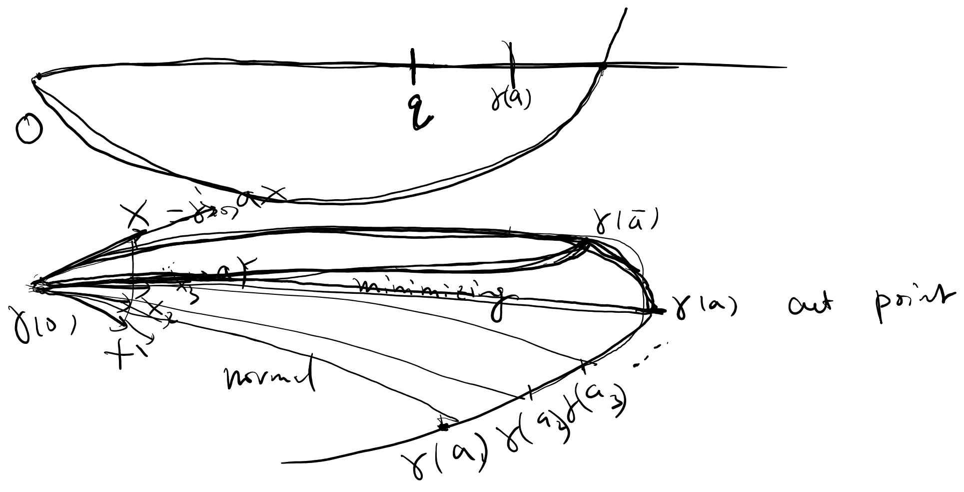
<u>Proof</u>:



Choose a sequence $a_1 > a_2 > a_3 > \cdots$ $\lim_{i \to \infty} a_i = a$

$$b_i := \underline{d(\gamma(0), \gamma(a_i))} < a_i$$

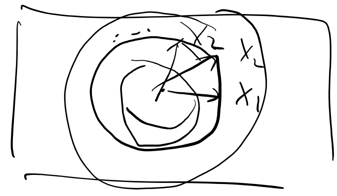
Pick $X_i \in T_{\gamma(0)}M$ s.t. $\gamma_i(t) = \exp_{\gamma(0)} tX_i$, $t \in [0, b_i]$

minimizing geodesic from $\gamma(0)$ to $\gamma(a_i)$

$\underline{X_i \neq \gamma'(0) = X}$, $\forall i = 1, 2, \dots$

$\gamma(a_i) = \exp_{\gamma(0)}(b_i X_i)$

$\{b_i X_i\}_{i=1}^{\infty}$

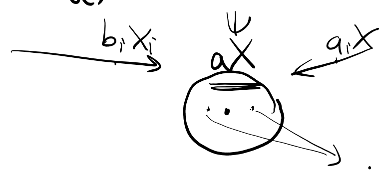


$T_{\gamma(0)}M$

$\lim_{i\to\infty} b_i = \lim_{i\to\infty} d(\gamma(0), \gamma(a_i)) = d(\gamma(0), \gamma(a)) = a$

$\boxed{\lim_{i\to\infty} b_i X_i =: aY,}$ Y unit vector.

① $Y = X$ ② $Y \neq X$ $\exp_{\gamma(0)} : T_{\gamma(0)}M \to M$



① If $Y = X$,

$\lim_{i\to\infty} b_i X_i = aY = aX = \lim_{i\to\infty} a_i X$.

$\exp_{\gamma(0)}(b_i X_i) = \gamma(a_i) = \exp_{\gamma(0)} a_i X$.

For any small neighborhood of aX, $\exp_{\gamma(0)}$ is not injective on it. $\rightsquigarrow$ aX is a critical point of $\exp_{\gamma(0)}$

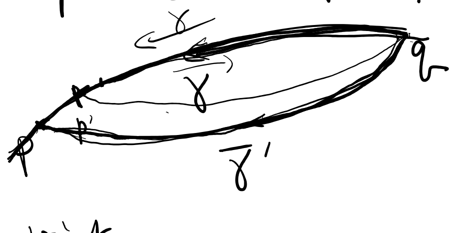
$\Rightarrow$ $\exp_{\gamma(0)} aX = \gamma(a)$ is a conjugate point.

$\rightsquigarrow$ 1^{st} conjugate point.

② If $Y \neq X$. $\exists$ two minimizing geodesics from $\gamma(0)$ to $\gamma(a)$. $\square$

Cor. In a complete Rie mfld M. If q is a cut point of along γ, then p is a cut pt of q along $\bar{\gamma}$

Proof: q is a cut pt of p along γ.



$\bar{\gamma}$. find q's cut point.

① q conjugate to p, $\Rightarrow$ p is conj to q $\Rightarrow$ p is a cut point of q

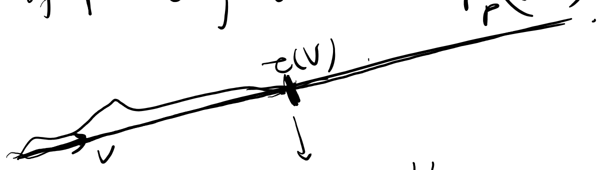
① q conjugate to p, ⇒ p is conj to q ⇒ p is a cut point of q

② ∃ 2 minimizing geodesic from p to q ⇒ p is a cut pt of q.

Thm 7. M complete manifold. given $p \in M$ □

$$E(p) := \{ tV : V \in T_pM, \ |V|_g = 1, \ 0 \le t < \tau(V) \}$$

$\tau(V)$ is the number s.t. $\exp_p(\tau(V)(V))$ is the cut point of p along $\gamma(t) = \exp_p(tV)$.

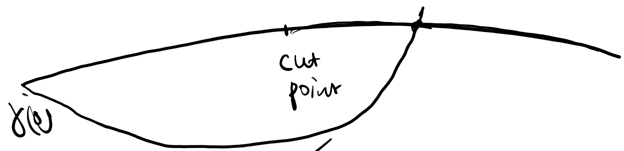


Then: $\exp_p$ maps $E(p)$ diffeomorphically onto a set of M.

and M is the union of $\exp_p(E(p))$, and $C(p)$.

Remark: $C(p)$ closed set. $\exp_p(E(p))$ open set.

Proof:



① $\exp_p(E_p) \cup C(p) = M$ ✓

∀ $q \in M$, ∃ minimizing geodesic $\gamma(t) = \exp_p tV$ from $\gamma(0) = p$ to $\gamma(a) = q$

⇒ $a \le \tau(V)$

So q is either a cut point, $q \in C(p)$ or $aV \in E_p$ ~~$q \in \exp_p$~~, $q \in \exp_p(E_p)$.

② $\exp_p(E_p) \cap C(p) = \varnothing$.

If not, ∃ $W \in E_p$ and ∃ $V \in \partial E_p$, $\boxed{W \ne V}$

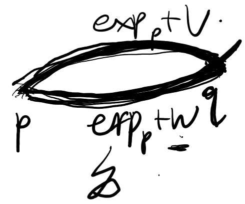
$\exp_p W = \exp_p V = q \in M$.

$\boxed{\partial E_p = \{ \tau(W) W : W \in T_pM, |W| = 1 \}}$

$\exp_p W = \exp_p V = q \in M.$ $\quad \left[W \in T_pM, \ |W|=1 \right\}$

If. $|V| \le |W|$, $\exp_p tV$, $t \in [0,1]$

$|V| \not< |W|$ $\quad \exp_p tW$, $t \in [0,1]$



If otherwise $\boxed{|V| > |W|}$ ↯

contradiction.

③ $\exp_p : E(p) \to \exp_p(E(p))$ is a diffeomorphism.

$\forall V \in E(p)$

$\boxed{(d\exp_p)_{(V)}} : T_V(T_pM) \longrightarrow T_{\exp_p(V)}M$ non-degenerate

Inverse fct thm $\Rightarrow$ $\exp_p$: local diffeomorphism.

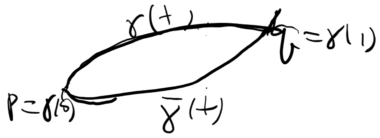
Remain: $\exp_p$ is injective.

$\forall w_1, w_2 \in E(p)$, $\exp_p(w_1) \neq \exp_p(w_2)$

If $\exists w_1, w_2 \in E(p)$, $w_1 \neq w_2$ s.t. $\exp_p(w_1) = \exp_p(w_2) = q$

$\gamma(t) = \exp_p(tw_1)$ $t \in [0,1]$

$\bar{\gamma}(t) = \exp_p(tw_2)$



minimizing

contradiction □

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