Hessian $\longrightarrow$ Laplacian $\longrightarrow$ volume comparison

Theorem (Hessian Comparison). Let $M, \bar{M}$ be two Rie. manifolds of the same dimension n and let $\gamma: [0,b] \to M$, $\bar\gamma: [0,b] \to \bar M$ be two normal geodesics. Denote $p = \gamma(0)$, $\bar p = \bar\gamma(0)$, $\rho(\cdot) := d(p, \cdot)$, $\bar\rho(\cdot) = \bar d(\bar p, \cdot)$ $(*)$

γ minimizing: $\rho \circ \gamma : [0,b] \to \mathbb{R}$, $t \mapsto t$

Suppose (1) $\gamma|_{[0,b]}$ and $\bar\gamma|_{[0,b]}$ are minimizing and $\boxed{\text{contain no cut point.}}$

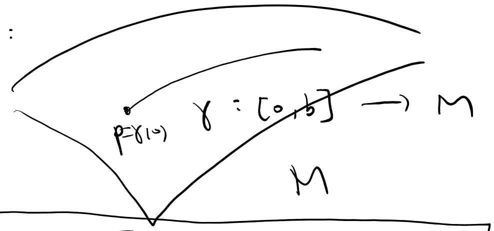
(2) $K(\Pi_{\gamma(t)}) \le \bar K(\bar\Pi_{\bar\gamma(t)})$ for all $t \in [0,b]$, for all 2-dim sections $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\bar\Pi_{\bar\gamma(t)} \subset T_{\bar\gamma(t)}\bar M$ containing $\dot\gamma(t)$, $\dot{\bar\gamma}(t)$ respectively.

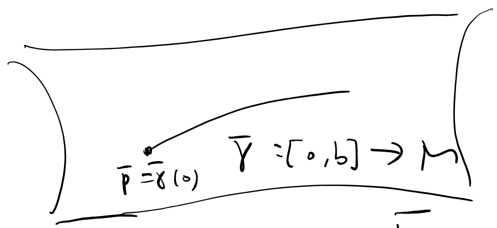
$\dot\gamma(t)$, $\gamma(t)$

Then we have $\boxed{\rho, \bar\rho \text{ are } C^\infty \text{ in a neighborhood}}$ of $\gamma|_{[0,b]}$ and $\bar\gamma|_{[0,b]}$ (except $p, \bar p$) respectively, and

$$\boxed{\operatorname{Hess}\rho \ge \operatorname{Hess}\bar\rho \quad \text{along } \gamma, \bar\gamma}$$

Remark:





$\bar M$

$\boxed{\operatorname{Hess}\rho \ge \operatorname{Hess}\bar\rho \quad \text{along } \gamma, \bar\gamma}$: $\forall\, t \in [0,b]$, $\forall\, X \in T_{\gamma(t)}M$, $\bar X \in T_{\bar\gamma(t)}\bar M$ such that $\boxed{|X|_g = |\bar X|_{\bar g} \quad \langle X, \dot\gamma(t)\rangle_g = \langle \bar X, \dot{\bar\gamma}(t)\rangle_{\bar g}}$

$$\operatorname{Hess}\rho\,(X, X) \ge \operatorname{Hess}\bar\rho\,(\bar X, \bar X)$$

At the point $\gamma(t)$, $\operatorname{Hess}\rho : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}$ symmetric, bilinear form

$\dot\gamma(t) \in T_{\gamma(t)}M$

$$\operatorname{Hess}\rho\,(\dot\gamma(t), \dot\gamma(t)) = \nabla^2(\rho)\,(\dot\gamma(t), \dot\gamma(t)) = \nabla_{\dot\gamma(t)}(\nabla\rho)\,(\dot\gamma(t))$$

$$= \nabla_{\dot\gamma(t)}\Big((\nabla\rho)(\dot\gamma(t))\Big) - (\nabla\rho)\big(\underbrace{\nabla_{\dot\gamma(t)}\dot\gamma(t)}_{=0}\big)$$

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$$= \nabla_{\dot\gamma(t)}\big((\nabla\rho)(\dot\gamma(t))\big) - (\nabla\rho)\big(\underbrace{\nabla_{\dot\gamma(t)}\dot\gamma(t)}_{=0}\big)$$

$$= \nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)}\rho = \frac{d^2}{dt^2}\underbrace{\rho\circ\gamma}_{\text{linear}} = 0$$

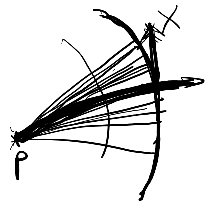
$\mathrm{Hess}\,\rho(\dot\gamma(t), X)$ $\quad X \in T_{\gamma(t)}M,\ \langle X, \dot\gamma(t)\rangle = 0$

$$= \nabla^2\rho(\dot\gamma(t), X)$$

$\rho\circ\gamma(t) = t$

$$= \underbrace{\nabla_X \big(\underbrace{\nabla_{\dot\gamma(t)}\rho}_{\equiv 1}\big)}_{=0} - \nabla_{\nabla_X\dot\gamma(t)}\rho$$

$\frac{d}{dt}\rho\circ\gamma \equiv 1$



$$= -\nabla_X\dot\gamma(t)(\rho) = -\langle \nabla_X\dot\gamma(t), \overset{\dot\gamma(t)}{\mathrm{grad}\,\rho}\rangle$$

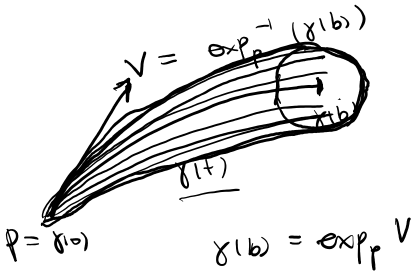
$$\langle \mathrm{grad}\,\rho, \dot\gamma(t)\rangle = \dot\gamma(t)\cdot\rho = \frac{d}{dt}(\rho\circ\gamma) = 1 \Rightarrow \mathrm{grad}\,\rho = \dot\gamma(t)$$

$$= -\langle \nabla_X\dot\gamma(t), \dot\gamma(t)\rangle = -\frac{1}{2}X\langle\dot\gamma(t),\dot\gamma(t)\rangle = 0$$

$$\Rightarrow \forall X \in T_{\gamma(t)}M,\quad \mathrm{Hess}\,\rho(X,X) = \mathrm{Hess}\,\rho(X^\perp, X^\perp)$$

$$X = \langle X, \dot\gamma(t)\rangle\dot\gamma(t) + X^\perp$$

Proof:



$\gamma(b) = \exp_p v$

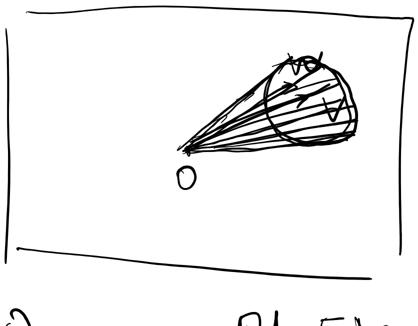
$E(p) = \{W \in T_pM$ s.t. $\exp_p W$ is not a cut pt of p along $\gamma(t) = \exp_p tW\}$

$V \in E(p)$ $\quad\exists\ \varepsilon$ small s.t.

$\forall\ W \in T_pM,\ \|W - V\| < \varepsilon$, we have $W \in E(p)$

T_pM.

$$U := \{W \in T_pM,\ \|W - V\| < \varepsilon\} \subset E(p)$$



$$\mathcal{U} := \{tW : t \in [0,1],\ W \in U\}$$

$$\exp_p\mathcal{U} \subset M$$

On $\exp_p\mathcal{U} \subset M$, $\rho(x) := |\exp_p^{-1}(x)|_g \quad \forall x \in \exp_p\mathcal{U}$

On $\exp_p \mathcal{U} \subset M$, $\underline{\rho(x)} := |\exp_p^{-1}(x)|_g \quad \forall x \in \exp_p \mathcal{U}$

ρ is C^∞ on $\exp_p \mathcal{U} \setminus \{p\}$

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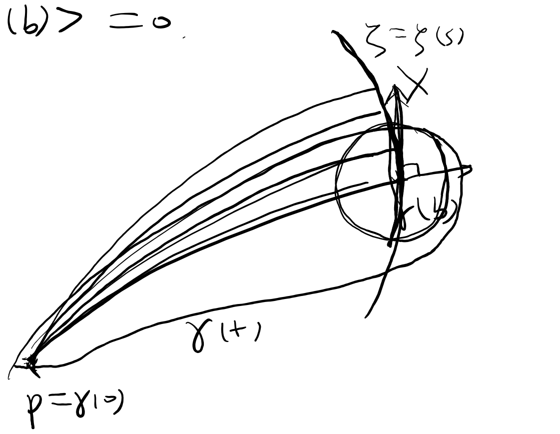
W.l.o.g., at $t=b$, $\forall X \in T_{\gamma(b)}M$, $\bar X \in T_{\bar\gamma(b)}\bar M$, s.t.

$$\langle X, \dot\gamma(b)\rangle = 0, \quad \langle \bar X, \dot{\bar\gamma}(b)\rangle = 0.$$

$$|X|_g = |\bar X|_{\bar g}$$



$\underline{\mathrm{Hess}\,\rho(X,X)}$, $\quad \overline{\mathrm{Hess}}\,\bar\rho(\bar X,\bar X)$

Index form

$\zeta = \zeta(s)$ geodesic s.t. $\zeta(0)=\gamma(b)$,
$\dot\zeta(0) = X$.

$\Rightarrow F : [0,b]\times(-\varepsilon,\varepsilon) \to M$ minimizing geodesic from $\gamma_s(0)=p$, $\gamma_s(b)=\zeta(s)$

$(t, s) \longmapsto F(t,s) := \gamma_s(t)$

$$\underline{F(t,s) := \underbrace{\exp_p\left(\tfrac{t}{b}\exp_p^{-1}(\zeta(s))\right)}_{C^\infty}}, \quad t\in[0,b],\ s\in(-\varepsilon,\varepsilon).$$

$U(t) = \frac{\partial}{\partial s}\Big|_{s=0} F(t,s)$ normal Jacobi field.

$$U(0)=0, \quad U(b) = \frac{\partial}{\partial s}\Big|_{s=0} F(t,s)\Big|_{t=b} = \frac{\partial}{\partial s}\Big|_{s=0} F(b,s)$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(\exp_p^{-1}(\zeta(s))) = \frac{\partial}{\partial s}\Big|_{s=0}\zeta(s)$$

$$= X$$

$$\mathrm{Hess}\,\rho(X,X) = \frac{d^2}{ds^2}\Big|_{s=0} \rho\circ\zeta(s) = L''(0) \qquad U = U^\perp$$

$$= \underbrace{\langle \underset{0}{\nabla_U U}, \dot\gamma\rangle\Big|_0^b}_{=0} + \underbrace{\int_0^b \langle\nabla_T U^\perp, \nabla_T U^\perp\rangle - \langle R(U^\perp,T)T,U^\perp\rangle\, dt}_{I_0^b(U,U)}$$

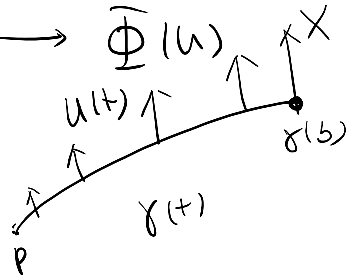
$$= I_0^b(U,U)$$

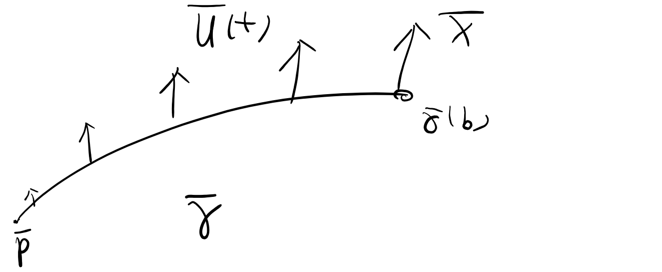
Similarly $\overline{\mathrm{Hess}}\,\bar\rho(\bar X,\bar X) = \bar I_0^b(\bar U,\bar U)$, $\quad \bar U(0)=0$, $\bar U(b)=\bar X$.

Similarly, $\overline{\text{Hess}\,\rho}\,(\bar{X},\bar{X}) = \bar{I}_0^b(\bar{U},\bar{U})$, $\bar{U}(0)=0$, $\bar{U}(b)=\bar{X}$.

? $\quad I_0^b(U,U) \overset{?}{\geq} \bar{I}_0^b(\bar{U},\bar{U})$

$U \longrightarrow \Phi(U)$





$\phi_b : T_{\gamma(b)}M \to T_{\bar{\gamma}(b)}\bar{M}$ preserve inner product

$$\underline{X} \longmapsto \bar{X}$$
$$\dot{\gamma}(b) \longmapsto \dot{\bar{\gamma}}(b) \hookrightarrow \Phi(U)(0) = 0, \quad \Phi(U)(b) = \bar{X}$$
$$\Phi(U)(0) = \bar{U}(0), \quad \Phi(U)(b) = \bar{U}(b)$$

$$\underset{\text{Hess}\,\rho(X,X)}{\underset{\|}{I(U,U)}} \underset{\text{curv. comparison condition}}{\geq} \bar{I}(\Phi(U), \Phi(U)) \underset{\text{minimizing property of Jacobi fields}}{\geq} \bar{I}(\bar{U},\bar{U}) = \overline{\text{Hess}\,\rho}\,(\bar{X},\bar{X})$$

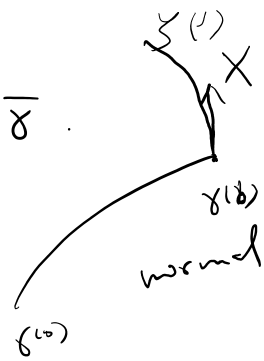
$\square$

<u>Cor.</u> Under the same assumption of the above Thm.

Let : $f : \underline{[0,b]} \longrightarrow \mathbb{R}$ C^∞ function, $f' \geq 0$.

Then we have

$$\text{Hess}\, f(\rho) \geq \overline{\text{Hess}}\, f(\bar{\rho}) \quad \text{along } \gamma, \bar{\gamma}.$$



<u>Proof</u>: $\text{Hess}\, f(\rho)(X,X) = \left.\frac{d^2}{ds^2}\right|_{s=0} f(\rho)(\zeta(s))$

$$= \left.\frac{d}{ds}\right|_{s=0}\left(f'(\rho(\zeta(s)) \cdot \left.\frac{d}{ds}\right|_{s=0} \rho(\zeta(s))\right)$$

$$= f''(\rho(\zeta(0)))\underline{\left(\left.\frac{d}{ds}\right|_{s=0}\rho(\underline{\zeta(s)})\right)^2} + f'(\rho(\zeta(s))) \cdot \left.\frac{d^2}{ds^2}\right|_{s=0}\rho(\zeta(s))$$

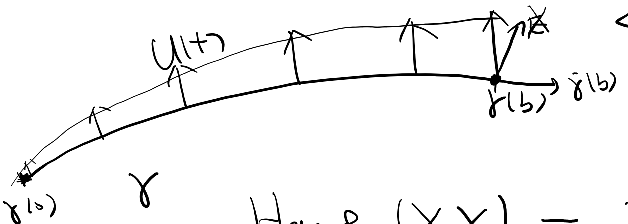
$$= \underbrace{\underbrace{f''(\rho(\zeta(0)))}_{= b}\underbrace{\langle X, \text{grad}\,\rho\rangle^2}_{= \langle \bar{X}, \dot{\bar{\gamma}}(t)\rangle}}_{} + \underbrace{\underbrace{f'(\rho(\zeta(s)))}_{\geq 0}\underbrace{\text{Hess}\,\rho(X,X)}_{\geq \overline{\text{Hess}}\,\bar{\rho}(\bar{X},\bar{X})}}_{}$$

$$\textcircled{=} \geq \overline{\text{Hess}}\, f(\bar{\rho})(\bar{X},\bar{X}).$$

$\square$

$$\Phi \geq \overline{\mathrm{Hess}\, f(\bar p)(\bar X, \bar X)}. \qquad \square$$

Example: Complete, simply-connected Rie. mfld with constant sectional curvature.



$\langle X, \dot\gamma(b)\rangle = 0$

no cut

$$\mathrm{Hess}\,\rho(X,X) = I(\overbrace{U}^{\text{normal Jacobi}}, U) = \int_0^b \langle \nabla_T U, \nabla_T U\rangle - \langle R(U,T)T, U\rangle\, dt$$

$\frac{d}{dt}\langle \nabla_T U, U\rangle - \langle \nabla_T \nabla_T U, U\rangle$

$$= \langle \nabla_T U, U\rangle \Big|_0^b = \langle \nabla_{\underline{T}} U, \underline{\underline{U}}\rangle (b)$$

$\{\underset{\overset{\|}{E_1}}{\dot\gamma(b)}, E_2, \cdots, E_n\}$ orthonormal basis of $T_{\gamma(b)}M$

parallel $\{\dot\gamma(t), E_2(t), \dots, E_n(t)\}$ orthonormal frame field along γ

$U(t) = f(t) E_i(t) \qquad i = 2, \dots, n,$

$\sum_i c_i \left(f(t) \underline{E_i(t)}\right)$

$$\nabla_T \nabla_T U + R(U,T)T = 0$$

$$\langle f''(t) E_i(t) + f(t) R(E_i, T)T, E_i\rangle = 0$$

$$f''(t) + f(t) \underbrace{K(E_i, T)}_{=k} = 0$$

$$\begin{cases} f''(t) + kf(t) = 0 \\ f(0) = 0, \end{cases} \Rightarrow f(t) = \begin{cases} \underline{\underline{C}}\,t & k = 0 \\ C \sin(\sqrt{k}\, t) & k > 0 \\ C \sinh(\sqrt{-k}\, t) & k < 0 \end{cases}$$

$U(t)$ s.t. $U(0) = 0$, $U(b) = X = f(b) E(b)$

$$\boxed{U(t) = \underline{f(t)} \cdot E(t),} \qquad E(b) = \frac{X(b)}{|X(b)|}$$

Jacobi equation
$$\begin{cases} f''(t) + kf(t) = 0 \\ f(0) = 0, \quad \underline{\underline{f(b)}} = |X(b)|. \end{cases}$$

$$\mathrm{Hess}\,\rho(X,X) = I(U,U) = \langle \nabla_T U(b), U(b)\rangle$$

$$= \langle \nabla_T U(b), X \rangle = \langle f'(b) E(b), X \rangle$$

$$= f'(b)|X| = \frac{f'(b)}{|X|}|X|^2 = \frac{f'(b)}{f(b)}|X|^2$$

$$= \frac{f_k'(b)}{f_k(b)} g(X, X)$$

$$\Rightarrow \quad \mathrm{Hess}\,\rho\,(X,X) = \frac{f_k'(b)}{f_k(b)} g(X,X)$$

In particular: ① $k=0$ $\quad \frac{f'(b)}{f(b)} = \frac{c}{c\cdot b} = \frac{1}{b}$

$$\mathrm{Hess}\,\rho\,(X,X) = \frac{1}{\rho} g(X,X) \qquad \langle X, \dot\gamma \rangle = 0$$

$X \in T_{\gamma(b)} M$

② $k=0$, $\quad \mathrm{Hess}\,\rho^2(X,X) = 2(X\rho)^2 + 2\rho\,\mathrm{Hess}\,\rho\,(X,X)$

$$= 2\langle X, \dot\gamma(b)\rangle^2 + 2\rho \cdot \frac{1}{\rho} g(X,X)$$

$$= 2g(X,X)$$

$$\Rightarrow \mathrm{Hess}\,\rho^2(X,X) = 2g(X,X), \quad \forall X,\ \langle X, \dot\gamma\rangle = 0$$

$\sec \le 0 \Rightarrow \mathrm{Hess}\,\rho^2 \ge 2g(X,X)$

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Cor: Let M be a complete simply-connected Rie mfld with $\sec \le 0$, $\rho = d(p, \cdot)$, $p \in M$.

Then $M \setminus \{p\}$ we have

$$\mathrm{Hess}\,\rho^2 \ge 2g$$

$$\mathrm{Hess}\,\rho \ge \frac{1}{\rho} g$$

trace $\quad \boxed{\Delta\rho \ge \frac{n-1}{\rho}}$

Laplacian Comparison:

Ricci lower bound $\Rightarrow$ $\Delta\rho$ upper bound

Thm. (Laplacian Comparison)

$(*)$ $\quad M^n, \overline{M}^n, \quad \gamma, \bar\gamma, \quad p, \bar p, \quad \rho, \bar\rho$

Suppose. (1) $\gamma|_{[0,b]}$, $\bar\gamma|_{[0,b]}$ minimizing, no cut point.

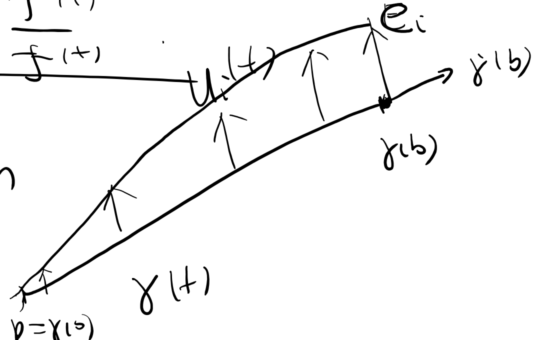
(2) $\forall t \in [0,b]$ $\quad \underline{Ric(\dot\gamma,\dot\gamma)(t)} \le \underline{\overline{Ric}(\dot{\bar\gamma},\dot{\bar\gamma})(t)}$ $\quad (n-1)k$

(3) $\overline{M}$ is a space form of sec curvature k

Then we have $\quad \Delta\rho(\gamma(t)) \le \bar\Delta\bar\rho(\bar\gamma(t)), \ \forall t \in (0,b]$.

~~Proof~~. Rmk: On a space form of sec $= k$,

$$\Delta\rho(\gamma(t)) = \frac{f'(t)}{f(t)}$$



Proof: $\{\overset{\dot\gamma(b)}{\overset{\shortparallel}{e_1}}, e_2, \dots, e_n\} \subset T_{\gamma(b)}M$ orthonormal.

$$\Delta\rho(\gamma(b)) := \sum_{i=2}^{n} \underbrace{Hess\,\rho(e_i,e_i)}$$

$Hess\,\rho(e_i,e_i) = I(U_i,U_i)$, where U_i is a Jacobi field along γ, $U_i(0)=0$, $U_i(b)=e_i$

$$\phi_b : T_{\gamma(b)}M \to T_{\bar\gamma(b)}\overline{M}$$
$$e_i \mapsto \bar e_i$$

$\rightsquigarrow \Phi$

$$\Phi(U_i)(0) = 0 = \bar U_i(0). \quad \Phi(U_i)(b) = \phi_b(e_i) = \bar e_i = \bar U_i(b)$$

$$\underline{\bar I(\Phi(U_i),\Phi(U_i)) \ge \bar I(\bar U_i,\bar U_i)} \quad \forall\, i=2,\dots,n$$

"$=$" $\Leftrightarrow$ $\Phi(U_i) = \bar U_i$ $\quad f e_i$

$$\sum_{i=2}^{n} I(U_i,U_i) \overset{?}{\ge} \sum_{i=2}^{n} \bar I(\Phi(U_i),\Phi(U_i)) \ge \sum_{i=2}^{n} \bar I(\bar U_i,\bar U_i)$$

$f'' + kf = 0$ $\qquad \boxed{\Phi(fe_i) = f\bar e_i}$

$$\sum_{i=2}^{n} \int_0^b \left(\langle \nabla_T U_i, \nabla_T U_i\rangle - \langle R(U_i,T)T, U_i\rangle\right) dt$$

$\shortparallel$

$$\ge \sum_{i=2}^{n} \int_0^b \langle \bar\nabla_{\bar T}\Phi(U_i), \bar\nabla_{\bar T}\Phi(U_i)\rangle - \langle \bar R(\Phi(U_i),\bar T)\bar T, \Phi(U_i)\rangle dt$$

$$\geq \sum_{i=2}^{n} \int_0^b \langle \nabla_T \bar\Phi(U_i), \nabla_T \bar\Phi(U_i) \rangle - \langle \bar R(\bar\Phi(U_i), \bar T)\bar T, \bar\Phi(U_i)\rangle \, dt$$

$$\Longleftrightarrow \boxed{\sum_{i=2}^{n} \int_0^b \langle R(\underbrace{U_i}_{f_k e_i}, T)T, \underbrace{U_i}_{f_k e_i}\rangle \, dt \leq \sum_{i=2}^{n} \int_0^b \langle \bar R(\bar\Phi(U_i), \bar T)\bar T, \bar\Phi(U_i)\rangle \, dt}$$

$$\langle R(U_i,T)T, U_i\rangle = K(U_i, T) \qquad \langle \bar R(\bar\Phi(U_i),\bar T)\bar T, \bar\Phi(U_i)\rangle = \bar K(\bar\Phi(U_i), \bar T)$$

$$Ric(T) = \sum_{i=2}^{n} K(E_i, T), \quad \{T, E_2, \dots, E_n\} \text{ orthonormal.}$$

$\{U_i,\ i=2,\dots,n,\ T\}(t)$, $\forall t \in (0,b]$ orthonormal?

$t=b$: $\{T(b), U_2(b), \dots, U_n(b)\} = \{e_1, \dots, e_n\}$ orthonormal

$\boxed{(3)\ M \text{ space form } sec = k.}$

$\{e_1, \dots, e_n\} \xrightarrow{\text{parallel}} \{e_1(t), \dots, e_n(t)\}$, $e_1(t) = \dot\gamma(t)$

$i=2,\dots,n$, $U_i(0)=0$, $U_i(b) = e_i$ $\qquad U_i = f_k(t)\, e_i(t)$

Jacobi field. $\begin{cases} f_k''(t) + k f_k(t) = 0 \\ f_k(0)=0, \quad f_k(b)=1 \end{cases}$

$\{T(t), U_2(t), \dots, U_n(t)\}$, $U_2(t) = f_k e_2(t)$, $U_n(t) = f_k e_n(t)$

$$\boxed{\bar\Phi(f_k(t) e_i(t)) = f_k(t)\bar e_i(t)}$$

$$\sum_{i=2}^{n} \int_0^b f_k^2 \langle R(e_i,T)T, e_i\rangle \, dt \leq \sum_{i=2}^{n} \int_0^b \langle \bar R(\bar\Phi(f_k e_i), \bar T)\bar T, \bar\Phi(f_k e_i)\rangle \, dt$$

$$= \sum_{i=2}^{n} \int_0^b f_k^2 \langle \bar R(\bar e_i, \bar T)\bar T, \bar e_i\rangle \, dt$$

$$\int_0^b f_k^2 \, Ric(T,T) \, dt \leq \int_0^b f_k^2 \, \underbrace{\overline{Ric}(\bar T, \bar T)}_{0} \, dt$$

<u>Remk</u>. "=" holds in Laplacian comparison

i.e. $\Delta\rho(\gamma(t)) = \bar\Delta\bar\rho(\bar\gamma(t))$, $\forall t \in (0,b]$

iff $\forall t \in (0,b]$ any section in $T_{\gamma(t)}M$ containing $\dot\gamma(t)$

i.e. [illegible]

iff $\forall t \in (0,b]$, any section in $T_{\bar\gamma(t)}\bar M$ containing $\dot{\bar\gamma}(t)$ has sectional curvature k, and any normal Jacobi field $\bar U(t)$ along $\bar\gamma$ with $\bar U(0)=0$ can be represented

$$\bar U(t) = f(t)\,E(t)$$

where $E(t)$ is a parallel v.f. along γ, $f:[0,b]\to\mathbb{R}$ C^∞ fct s.t. $f''+kf=0$, $f(0)=0$.

Cor. Under the same assumption of Laplacian comparison. Let $f:[0,b]\to\mathbb{R}$ C^∞ fct s.t. $f'\geq 0$.

Then. $$\Delta f(\rho)(\gamma(t)) \leq \bar\Delta f(\bar\rho)(\bar\gamma(t)), \quad \forall t\in(0,b].$$

Proof:

Hess $f(\rho)$

$$\rightarrow \quad \boxed{\Delta(f(\rho))(\gamma(t)) = f''(\rho) + f'(\rho)\,\Delta\rho.}$$

$$\bar\Delta(f(\bar\rho))(\bar\gamma(t)) = f''(\bar\rho) + f'(\bar\rho)\,\bar\Delta\bar\rho$$

Exercise. □

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