2021-2022 ~~Autumn~~ Fall "Selected topics in Geometric Analysis".

Geometric Analysis on graphs

Introduction:

On June 10, 1854, Riemann delivered his probationary inaugural lecture for seeking the position of "Privatdocent" to the Faculty of Götingen University, which is entitled

[1] Riemann, "Über die Hypothesen, welche der Geometrie zu Grund liegen" (On the hypotheses which lie at the fundation of geometry), 1854, published later in 1866.

In the beginning, Riemann mentioned, "As is well known, geometry presupposes the concept of space".

What is a space? This is a natural and important question. We live in a "space". There are many "object's in the space. You feel that some objects are moving, and the others are stay put. We can have lots of empirical experence.

We study "objects" abstractly in mathematics. If we identify different objects, we ~~will~~ then obtain numbers/ quantities from a set of objects.

Riemann [1]: "Notions of quantity are possible only when there already exists a general concepts which admits particular instances. These instances forms either a continuous or a discrete manifold, depending on whether or not a continuous transition of instances can be found between any two of them."

"Their quantitative comparison is effected in the case of discrete quantities by counting. in the case of ~~quan~~ continuous quantities by measurement."

Counting leads to integers, while measurement to reals. Therefore, measurement is not only "counting".

[2] Jost, Mathematical concepts, Springer, 2015.
[Section 1.1.3].

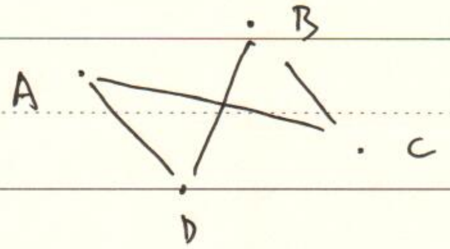
Discrete spaces has many examples in reality: the "space" of all chairs for example. Continuous space also has many instances: the "space" of all colors for example.

Riemann [1]: "... the theorems of geometry cannot be deduced from general notions of quantities, but that those properties which distinguish space from other conceivable triply extended quantities can only be deduced from experience."

What is the "experience" that distinguish ~~quantities~~ spaces from quantities? Leibniz (1646-1716) has a very deep view: "Space simply encode relations between objects instead of existing independently of the objects contained in it." (Jost [2], pp. 125).

"Relations" can ~~have~~ be of different types: (Jost [2], pp. 6)

(i) Discrete relations.



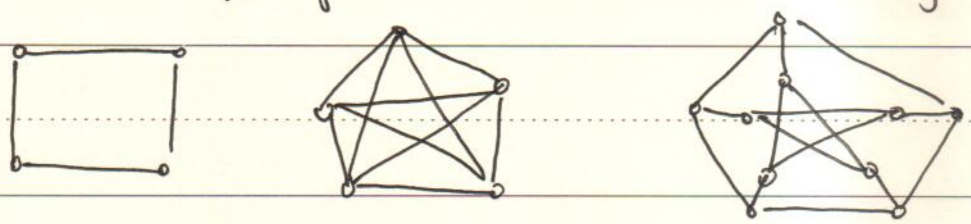
For example, we have objects A, B, C, D. If ~~they~~ 2 of them have relations, we put an edge. If 2 of them have no relations, we do nothing.

(ii) Qualitative relations: For example, nearness (→ topological space)

(iii) Quantitative: For example, distance between two objects.

Typical discrete spaces are represented by "graphs".

A graph $G = (V, E)$ is a set V of vertices together with a set E of edges, which is a set of vertex-pairs.



In this course, we discuss geometric analysis problems on graphs.

There are two outstanding landmarks in the history of graph theory. One is Euler's solution of "Königsberg Bridge Problem", dated 1736, and the other is the appearance of Dénes König's textbook in 1936."

[3] König, Theory of finite and Infinite graphs,
[Theorie der endlichen und unendlichen Graphen]
Akademische Verlagsgesellschaft Leipzig 1936.

"From Königsberg to König's book". So runs the graphic tale ...".

König's book presenting Graph Theory to the mathematical world as a subject in its own right. (See Tutte, Commentary of König's book).

<u>History</u>.

1736 Euler : Königsberg Bridge Problem

<u>1852</u>. Four color Problem

"Is it true that any map drawn in the plane may have its regions colored with 4 colors, in such a way that any two regions have a common border have different colors?"

Letter from De Morgan to Hamilton. (as a question of Francis Guthrie).

1847. Kirchhoff : Electrical networks.

1857. Cayley : Trees and enumerative graph theory and applications in Chemistry.

1878. Sylvester: Graph theory and chemistry introducing the term "graph".

1931–1933. Hassler Whitney. Published several graph-theoretical papers.

1936. König's book published.

Let $G=(V,E)$ be a graph.

$E = \{\{u,v\} : u,v \in V\}$. We also write $u \sim v$ if $\{u,v\} \in E$.

We put some restrictions.

① $\{u,u\} \notin E$, $\forall u \in V$. That is, G does not have self-loops. We call such a graph "simple".

② The degree of $u \in V$, denoted by d_u is defined as

$$d_u = \sum_{v \in V,\, \{u,v\} \in E} 1 .$$

We require $d_u < \infty$, $\forall u$. We call such a graph "locally finite".

③ We require G to be "connected", i.e., $\forall u,v \in V$, there exists a path $u u_1 u_2 \cdots u_n v$ where

$$u \sim u_1 \sim u_2 \sim \cdots \sim u_n \sim v.$$

<u>Theorem</u> 1. (König, Chapter VI, §1) The set of vertices V of a connected, locally finite graph $G=(V,E)$ is finite or countably infinite.

<u>Proof</u>. We define the <u>combinatorial distance</u> $d(u,v)$ between any $u,v \in V$ to the length of the shortest path connecting them. Then pick any $u \in V$. Consider

$$S_0(u) = \{u\}$$

$$S_1(u) = \{v \in V : d(u,v)=1\}$$

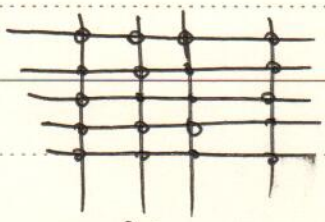
$$S_2(u) = \{v \in V : d(u,v)=2\}$$

$$\vdots$$

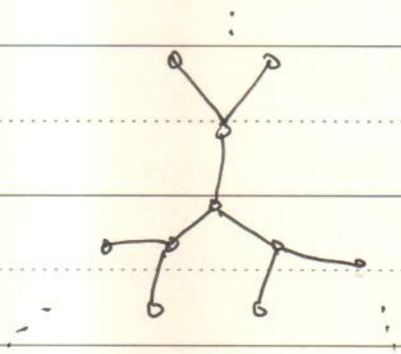
Then $V = \bigcup_{i=0}^{\infty} S_i(u)$ since G is connected.

The finiteness of degrees of each vertex tells that $S_i(u)$ is finite for any i. Therefore, $|V|$ is a sum of finitely or countably finite many finite sets. Hence, $|V|$ is finite or countably finite. $\square$

Notice that, (G, d) is a metric space. Due to ~~the~~ Gauss' principle of intrinsic geometry, a graph coupled with its combinatorial distance, has its own intrinsic geometry. We may talk about certain intrinsic "curvature" notions of graphs.



"flat".



A main ~~point~~ viewpoint in geometric analysis ~~is to~~ study the geometry of the underlying space is to study the <u>functions</u> on the space.

Our topics range from harmonic functions, Laplacian eigenvalues and eigenfunctions, heat kernels, etc.

(I) Harmonic functions and Laplacians.

Let's assume $G=(V,E)$ be a connected finite simple graph.

~~Suppose~~ Consider a function $f: V \to \mathbb{R}$. We only know its values on $W \subset V$. We'd like to guess the values of f at the vertices not in W.

[Daniel A. Spielman, Graphs and networks, Lecture 12]

Imagine that every edge is a spring. For each $v \in W$, we nail the vertex down onto the real line at $f(v)$. We then let the other rings settle into position, and guess that $f(u)$ is the location of u.

Assume that each spring ~~has~~ is ideal with spring constants 1. ~~Actually~~ For $\{u,v\} \in E$, Hooke's law tells that the forces it exerts at u is in the direction of v and is proportional to the distance between $f(u)$ and $f(v)$. That is, the force is

$$f(v) - f(u).$$

In a stable configuration, all of the vertices that have not been nailed down must experience a zero net force. That is.

$$(1) \qquad \sum_{v:\{u,v\}\in E} \big(f(v) - f(u)\big) = 0$$

$$\Rightarrow \frac{1}{d_u} \sum_{v:\{u,v\}\in E} f(v) = f(u).$$

That is, each vertex that is not nailed down is the average of its neighbors.

A function f that satisfies (1) for any $u \in V \setminus W$ is called <u>harmonic</u> on $V \setminus W$.

<u>Natural Questions</u> about solutions of (1): Existence? Uniqueness?

Physics tells us that the vertices will settle into the position that minimizes the <u>potential energy</u>. The potential energy of an ideal linear spring with constant w when stretched into length l is

$$\frac{1}{2} w l^2.$$

So, we have

$$E(f) := \frac{1}{2} \sum_{\{u,v\} \in E} (f(u) - f(v))^2$$

<u>Observation 1</u>. $E(f) \geq 0$. Therefore, E at least has <u>one</u> minimum.

<u>Observation 2</u>: $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(f + \varepsilon \delta_u)$

$$= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{1}{2} \sum_{v : \{u,v\} \in E} (f(u) + \varepsilon - f(v))^2$$

$$= \sum_{v : \{u,v\} \in E} (f(u) - f(v))$$

That is, any (local) minimizer of E is <u>harmonic</u>.

This shows the <u>existence</u> of the solution of (1).

<u>Theorem D.1</u>. Let $G=(V,E)$ be a finite, connected, simple graph. Let $W \subset V$ and $|W| \geq 1$. Then the minimizer of E is unique.

<u>Proof</u>. Suppose that we have 2 different minimizers of E, say, f and g. Take the average of them:

$$h := \frac{1}{2}(f+g).$$

Then $h|_W = f|_W = g|_W$.

We compute

$$2E(f) \leq 2E(h) = \cancel{\tfrac{1}{2}} \sum_{\{u,v\}\in E} (h(u)-h(v))^2$$

$$= \cancel{\tfrac{1}{2}} \sum_{\{u,v\}\in E} \left(\frac{1}{2}(f(u)-f(v)) + \frac{1}{2}(g(u)-g(v)) \right)^2$$

$$\leq \frac{1}{2} \sum_{\{u,v\}\in E} (f(u)-f(v))^2 + \frac{1}{2} \sum_{\{u,v\}\in E} (g(u)-g(v))^2$$

$$= E(f) + E(g) = 2E(f).$$

with "=" holds iff $f(u)-f(v) = g(u)-g(v)$, $\forall \{u,v\}\in E$.

$\left((a+b)^2 \leq 2a^2+2b^2, \text{ "=" iff } a=b \right)$

Therefore $f(u)-f(v) = g(u)-g(v)$, $\forall \{u,v\} \in E$.

Due to the connectivity of G, we have $f=g$. $\square$

However, this does not implies the uniquess of sol. of (1) immediately. We need the following:

Theorem 2. $|W|>1$. The only local minimizer of E is the global minimizer.

Proof. Let h be the global minimizer, and let f be any minimizer. Consider

$$f_\varepsilon := (1-\varepsilon)f + \varepsilon h$$

Then $f_\varepsilon|_W = f|_W = h|_W$.

We compute

$$\begin{aligned}2E(f_\varepsilon) &= \sum_{\{u,v\}\in E} (f_\varepsilon(u) - f_\varepsilon(v))^2 \\ &= \sum_{\{u,v\}\in E}\left[(1-\varepsilon)(f(u)-f(v))^2 + \varepsilon(h(u)-h(v))^2\right] \\ &\quad - \sum_{\{u,v\}\in E} \varepsilon(1-\varepsilon)\left[(f(u)-f(v))-(h(u)-h(v))\right]^2\end{aligned}$$

(We're using here $((1-\varepsilon)a+\varepsilon b)^2 = (1-\varepsilon)a^2+\varepsilon b^2 - \varepsilon(1-\varepsilon)(a-b)^2$).

Therefore

$$\begin{aligned}2E(f_\varepsilon) &= 2(1-\varepsilon)E(f) + 2\varepsilon E(h) \\ &\quad - \varepsilon(1-\varepsilon)\sum_{\{u,v\}\in E}\left[(f(u)-f(v))-(h(u)-h(v))\right]^2 \\ &\overset{\text{if } f \text{ is not a global minimizer}}{<} 2E(f) - \varepsilon(1-\varepsilon)\sum_{\{u,v\}\in E}\left[(f(u)-f(v))-(h(u)-h(v))\right]^2\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{E(f_\varepsilon)-E(f)}{\varepsilon} &\leq \lim_{\varepsilon\to 0} -\frac{1-\varepsilon}{2}\sum_{\{u,v\}\in E}\left[(f(u)-f(v))-(h(u)-h(v))\right]^2 \\ &= -\frac{1}{2}\sum_{\{u,v\}\in E}\left[(f(u)-f(v))-(h(u)-h(v))\right]^2 \\ &< 0 \quad \text{since } f\neq h.\end{aligned}$$

This contradicts to the fact $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(f_\varepsilon) = 0$. □

Then we have shown that the solution of (1) is unique.

In fact, the uniqueness can be proved more directly.

<u>Theorem 3</u> Let f and g be two functions that are both harmonic on $V\setminus W$ and $f|_W = g|_W$.
Then we have
$$f|_{V\setminus W} = g|_{V\setminus W}.$$
<u>Proof</u>: Set $h = f - g$. Then $h|_W = 0$.
Let v_0 be a maximum of h, i.e. $h(u) \leq h(v_0)$, $\forall u \in V$.
Suppose, $v_0 \in V\setminus W$. Then
$$h(v_0) = \frac{1}{d_{v_0}} \sum_{u:\{u,v_0\}\in E} h(u) \leq h(v_0)$$
$\Rightarrow$ " = " holds, which forces $h(u) = h(v_0)$, $\forall \{u, v_0\} \in E$.
By induction, $h(u) = h(v_0)$ for any $u \in V$ reachable from v_0. Pick $u = w \in W$. Then we have $h(v_0) = 0$. That is, h also achieves its maximal value on W.
Similarly, we show that h achieves its minimal value also on W. Therefore, we have $h \equiv 0$. □

<u>Remark</u> In this result, the graph G can be ~~only~~ infinite. (but locally finite)

<u>Laplacian operator</u>: $L: \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$
$\forall f: V \to \mathbb{R}$, $Lf \in \mathbb{R}^{|V|}$ s.t.
$$Lf(u) = -\frac{1}{d_u} \sum_{v:\{u,v\}\in E} (f(u) - f(v)).$$
Observe that a harmonic fct is a solution of
$$\begin{cases} Lf(u) = 0, & \forall u \in V\setminus W \\ f|_W \text{ is given} \end{cases}$$

$L: \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ is an analogue of the Laplace-Beltrami operator on Rie. mflds.

<u>Naive intuition</u>: Consider a lattice graph.