Hence, we obtain

$$2\sum_u f(u)^2 d_u - \sum_{\{u,v\}} (f(u)-f(v))^2 = \sum_u f(u)^2 d_u + \frac{1}{2}\sum_u \sum_{v: v\sim u} 2 f(u) f(v)$$

$$= \sum_{\{u,v\}\in E} (f(u)+f(v))^2 .$$

This completes the proof. □

<u>Proof of $2-\lambda_N \leq 2\beta$</u>:

Let $(V_1, V_2)$ be the pair s.t. $\frac{E(V_1)+E(V_2)+|\partial(V_1\cup V_2)|}{\mathrm{vol}(V_1\cup V_2)}=\beta$.

Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x\in V_1 \\ -1, & \text{if } x\in V_2 \\ 0, & \text{otherwise.} \end{cases}$$

Since $V_1\cup V_2 \neq \phi$, we know $f\neq 0$. Hence

$$2-\lambda_N \leq \frac{\sum_{\{u,v\}} (f(u)+f(v))^2}{\sum_u f(u)} = \frac{2E(V_1)+2E(V_2)+|\partial(V_1\cup V_2)|}{\mathrm{vol}(V_1\cup V_2)} \leq 2\beta.$$ □.

Next, we discuss the proof of $2-\lambda_N \geq \frac{\beta^2}{2}$.

We will make use of Lemma II.3.1 and adopt a very analogous strategy as in the proof of Cheeger ineq. For convenience, we denote

$$\bar{R}(f) = \frac{\sum_{\{u,v\}\in E} (f(u)+f(v))^2}{\sum_{u\in V} f(u)^2 d_u}$$

and

$$\psi(V_1,V_2) := \frac{E(V_1)+E(V_2)+|\partial(V_1\cup V_2)|}{\mathrm{vol}(V_1\cup V_2)}$$

We aim at relating $\bar{R}(f)$ and $\psi(V_1,V_2)$ for some $(V_1,V_2)$.

Let us recall what we did on page (74). By interchanging the order of integration / summation, we have

$$(\maltese)\quad \frac{\int_0^\infty |\partial\Omega_t|\,dt}{\int_0^\infty \mathrm{vol}(\Omega_t)\,dt} = \frac{\sum_{\{u,v\}\in E} |g(u)^2 - g(v)^2|}{\sum_{u\in V} g(u)^2\,du} \overset{\text{Cauchy-Schwarz}}{\leq} \sqrt{2R(g)}$$

where $\Omega_t = \{x \in V : g^2(x) > t\}$, $t \in [0,\infty)$.

If we define the indicator function

$$\chi_g^t(u) = \begin{cases} 1, & g^2(u) > t \\ 0, & \text{otherwise} \end{cases}$$

We can rewrite $|\partial\Omega_t| = \sum_{\{u,v\}\in E} |\chi_f^t(u) - \chi_f^t(v)|$

$$\mathrm{vol}(\Omega_t) = \sum_{u\in V} |\chi_f^t(u)|\,du.$$

Then $(\maltese)$ becomes

$$\frac{\int_0^\infty \sum_{\{u,v\}\in E} |\chi_f^t(u) - \chi_f^t(v)|\,dt}{\int_0^\infty \sout{\mathrm{vol}(\Omega_t)}\ \sum_{u\in V}|\chi_f^t(u)|\,du\ dt} \leq \frac{\sum_{\{u,v\}\in E} |g(u)-g(v)||g(u)+g(v)|}{\sum_{u\in V} g(u)^2\,du}$$

where $\phi(\Omega_t) = \dfrac{\sum_{\{u,v\}\in E} |\chi_f^t(u) - \chi_f^t(v)|}{\sum_{u\in V} |\chi_f^t(u)|\,du}$.

We will find a "indicator function" for $\psi$.

Given a nonzero function $f : V \to \mathbb{R}$, we define

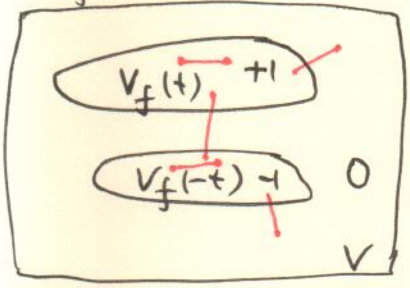
$$V_f(t) = \{u \in V : f(u) > \sqrt{t}\},\quad V_f(-t) = \{u\in V : f(u) < -\sqrt{t}\}$$

and

$$\bar{\chi}_f^t(u) = \begin{cases} 1, & \text{if } f(u) > \sqrt{t}, \text{ i.e. } u \in V_f(t) \\ -1, & \text{if } f(u) < -\sqrt{t}, \text{ i.e. } u \in V_f(-t) \\ 0, & \text{otherwise, i.e. if } |f(u)| \leq \sqrt{t}. \end{cases}$$

By definition, we check

$$\sum_{\{u,v\}\in E} |\bar{\chi}_f^t(u) + \bar{\chi}_f^t(v)| = E(V_f(t)) + E(V_f(-t)) + |\partial(V_f(t) \cup V_f(-t))|$$




and

$$\sum_{u\in V} |\bar{\chi}_f^t(u)|\,du = vol\left(V_f(t) \cup V_f(-t)\right)$$

That is, we have

<u>Lemma II.3.2</u>: $\psi\left(V_f(t), V_f(-t)\right) = \dfrac{\sum_{\{u,v\}\in E} |\bar{\chi}_f^t(u) + \chi_f^t(v)|}{\sum_{u\in V} |\chi_f^t(u)|\,du}$.

We furthe derive the following results:

<u>Lemma II.3.3</u>: $\int_0^\infty \sum_{u\in V} |\bar{\chi}_f^t(u)|\,du\,dt = \sum_{u\in V} f(u)^2\,du$.

<u>Proof</u>: $LHS = \int_0^\infty \sum_{\substack{u\in V\\ |f(u)|>\sqrt{t}}} 1\,du\,dt = \sum_{u\in V} du \cdot \int_0^{f(u)^2} dt = \sum_{u\in V} f(u)^2 du.$ □

<u>Lemma II.3.4</u>: $\int_0^\infty \sum_{\{u,v\}\in E} |\bar{\chi}_f^t(u) + \bar{\chi}_f^t(v)|\,dt$

$$\leqslant \sum_{\{u,v\}\in E} |f(u)+f(v)|\left(|f(u)| + |f(v)|\right)$$

<u>Proof</u>: For any given $\{u,v\}\in E$, W.l.o.g., we suppose $|f(u)| \geqslant |f(v)|$.

<u>Case 1</u>: $f(u)$ and $f(v)$ have different signs.

$$|\bar{\chi}_f^t(u) + \bar{\chi}_f^t(v)| = \begin{cases} 0 & , t < f(v)^2 \\ 1 & , f(v)^2 \leq t < f(u)^2 \\ 0 & f(u)^2 \leq t \end{cases}$$

Case 2. $f(u)$ and $f(v)$ have the same sign,

$$|\bar{\chi}_f^t(u) + \bar{\chi}_f^t(v)| = \begin{cases} 2 & , t < f(v)^2 \\ 1 & , f(v)^2 \leq t < f(u)^2 \\ 0 & , f(u)^2 \leq t \end{cases}$$

$$\int_0^\infty \sum_{\{u,v\}\in E} |\bar{X}_f^t(u) + \bar{X}_f^t(v)|\, dt$$

$$= \sum_{\{u,v\}\in E} \int_0^\infty |\bar{X}_f^t(u) + \bar{X}_f^t(v)|\, dt$$

In case 1, we have ( $f(u)$, $f(v)$ have different sign)

$$\int_0^\infty |\bar{X}_f^t(u) + \bar{X}_f^t(v)|\, dt = f(u)^2 - f(v)^2$$

$$= (f(u) - f(v))(f(u) + f(v))$$

$$\leq |f(u) + f(v)|(|f(u)| + |f(v)|)$$

In Case 2, $f(u)$, $f(v)$ have the same sign, we have

$$\int_0^\infty |\bar{X}_f^t(u) + \bar{X}_f^t(v)|\, dt = 2f(v)^2 + f(u)^2 - f(v)^2$$

$$= f(u)^2 + f(v)^2$$

$$\leq f(u)^2 + f(v)^2 + 2f(u)f(v)$$

$$= (f(u) + f(v))^2$$

$$= |f(u) + f(v)|(|f(u)| + |f(v)|).$$

Therefore,

$$\sum_{\{u,v\}\in E} \int_0^\infty |\bar{X}_f^t(u) + \bar{X}_f^t(v)|\, dt$$

$$\leq \sum_{\{u,v\}\in E} |f(u) + f(v)|(|f(u)| + |f(v)|). \qquad \square.$$

<u>Proof of $2 - \lambda_N \geq \frac{\beta^2}{2}$</u>: For any nonzero function $f$,

By Lemma (II.3.3) & (II.3.4), we have

$$\frac{\int_0^\infty \sum_{\{u,v\}\in E} |\bar{X}_f^t(u) + \bar{X}_f^t(v)|\, dt}{\int_0^\infty \sum_{u\in V} |\bar{X}_f^t(u)|\, du\, dt} \leq \frac{\sum_{\{u,v\}\in E} |f(u) + f(v)|(|f(u)| + |f(v)|)}{\sum_{u\in V} f(u)^2\, du}$$

$$\leq \frac{\sqrt{\sum_{\{u,v\}\in E} (f(u) + f(v))^2} \cdot \sqrt{\sum_{\{u,v\}\in E} (|f(u)| + |f(v)|)^2}}{\sum_{u\in V} f(u)^2\, du}.$$

Using $\sum_{\{u,v\}\in E} \left(|f(u)|+|f(v)|\right)^2 \le 2\sum_{\{u,v\}\in E}\left(f(u)^2+f(v)^2\right)$

$$= 2\cdot\frac{1}{2}\sum_{u\in V}\sum_{v: v\sim u}\left(f(u)^2+f(v)^2\right) = 2\sum_{u\in V} f(u)^2 d_u .$$

we have

$$\frac{\int_0^\infty \sum_{\{u,v\}\in E} |\bar{\chi}_f^t(u)+\bar{\chi}_f^t(v)|\,dt}{\int_0^\infty \sum_{u\in V} |\bar{\chi}_f^t(u)|\,d_u\,dt} \le \sqrt{2\bar{R}(f)} . \qquad (*)$$

Noticing Lemma (II.3.2), we know there exists a $t_0\in[0,\infty)$, s.t. $\psi\left(V_f(t_0), V_f(-t_0)\right) \le$ LHS of $(*)$.

Using Lemma II.3.1, we obtain

$$\text{RHS of } (*) \le \sqrt{2(2-\lambda_N)}$$

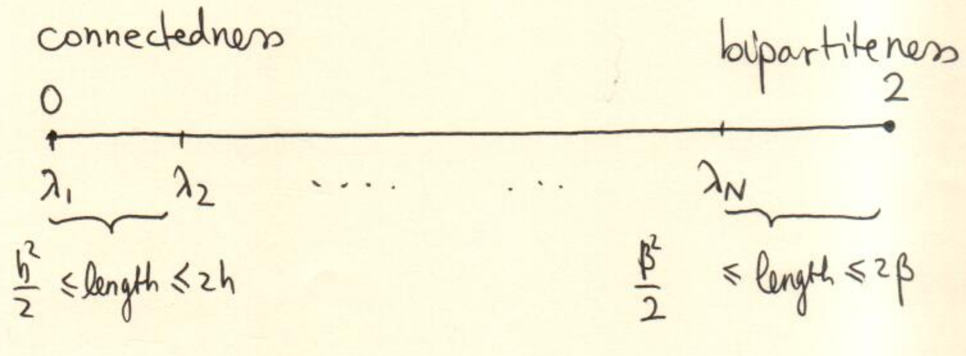
Therefore $(*)$ implies

$$\psi\left(V_f(t_0), V_f(-t_0)\right) \le \sqrt{2(2-\lambda_N)} .$$

Hence we have

$$2-\lambda_N \ge \frac{\beta^2}{2} .$$

□

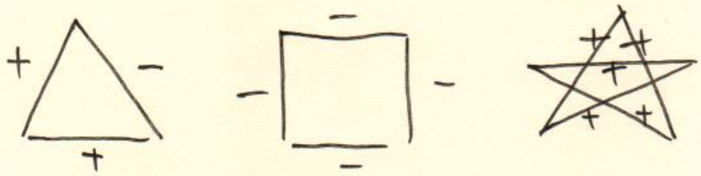



Although connectedness and bipartiteness are two quite different properties, we see great similarities in the proof the Cheeger and dual Cheeger inequalities.

## (II.4) Signed graphs and balancedness.

So our natural guess is that there must be ~~something~~ some hidden structure underlying both extremal eigenvalues and the connectedness and bipartiteness. This structure is the so-called "signed graph" introduced by Frank Harary in 1954.

A signed graph $\Gamma = (G, \sigma)$ is a graph $G = (V, E)$ coupled with a signature $\sigma : E \to \{+1, -1\}$.



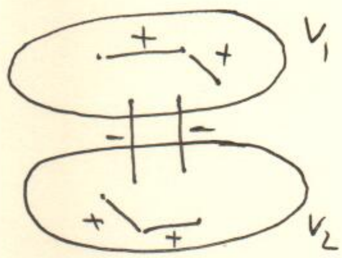

In some sense, this is to say, we distinguish a subsets of $E$ in $G = (V, E)$.

For convenience, we write $\sigma_{uv} := \sigma(\{u, v\})$. The sign of a cycle in $G$ is defined to be the product of the signs of all its edges.

<u>Definition</u>: [Harary] A signed graph $(G, \sigma)$ is called balanced if all cycles in $G$ are positive.

[Harary] : On the notion of balance of a signed graph, Michigan Math. J. 2 (1953), no. 2, 143-146.
1954

Theorem II.4.1 (Harary's balance Theorem) A signed graph $\Gamma = (G, \sigma)$ is balanced iff there exists a bipartition of the set $V$ into two disjoint $V_1$ and $V_2$ (one of which may be empty) such that, each positive edge connects 2 vertices of the same subset and each negative edge connects 2 vertices of different subsets.



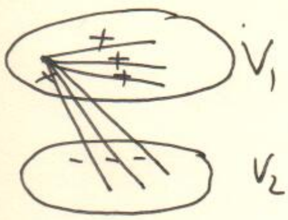

Proof of sufficiency ~~necessity~~ ($\Leftarrow$): Suppose that the conditions of the theorem are satisfied.

Since every cycle in $G$ contains an even number of edges connecting 2 vertices of different subsets, every cycle is $+$.

Proof of necessity: ($\Rightarrow$)

We first consider the special case that the underlying graph $G = (V, E)$ is complete.

Pick a vertex $x \in V$. Let us set




$$V_1 = \{x\} \cup \{y \in V, \ \sigma_{xy} = +1\}$$

$$V_2 = \{y \in V : \sigma_{xy} = -1\}$$

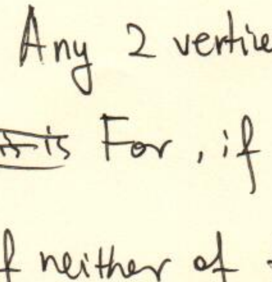
- Any 2 vertices $y_1, y_2$ of $V_1$ satisfy $\sigma_{y_1 y_2} = +1$. For, if one of them, say, $y_1 = x$, this is the construction of the set $V_1$. If neither of them is $x$, then the 3-cycle containing $x, y_1, y_2$ ~~contains~~ satisfies $\sigma_{xy_1} = \sigma_{xy_2} = +1$. Since this cycle is positive, we have $\sigma_{y_1y_2} = +1$.

Any 2 vertices $z_1, z_2$ of $V_2$ also satisfy $\sigma_{z_1z_2} = +1$. This is seen from the fact that the positive 3-cycle containing $x, z_1, z_2$ satisfies $\sigma_{xz_1} = \sigma_{xz_2} = -1$.

Any 2 vertices $w_1, w_2$ s.t. $w_1 \in V_1$, $w_2 \in V_2$ satisfy $\sigma_{w_1w_2} = -1$. ~~This is~~ For, if one of them is $x$, this is by construction.

If neither of them is $x$, the positive 3-cycle containing $x, w_1, w_2$ satisfies $\{\sigma_{xw_1} ; \sigma_{xw_2}\} = \{+1, -1\}$, which forces $\sigma_{w_1w_2} = -1$. □

For the general case, we only need show that any balanced signed graph can be extended to a complete signed graph. The potential difficulty lies in: For $u \neq v$, there ~~may~~ might




exist many paths connecting them. If 2 paths have different sign, then we do not have a ~~consistant of~~ consistent way to assign a sign to the new ~~ad~~ edge $\{u, v\}$ to keep the whole graph being balanced. However, ~~+~~ we have the following fact, which excludes such possibility.

<u>Prop</u>: Any signed graph $\Gamma = (G, \sigma)$ is balanced iff for each pair of distinct vertices $u, v$, all paths connecting $u$ and $v$ have the same sign.

<u>Proof</u>: $(\Leftarrow)$ Every cycle can be decomposed into 2 paths

connecting the same pair of distinct vertices. The sign of the cycle is the product of the signs of the 2 paths, and therefore, every cycle is positive.

($\Leftarrow$). Let $\alpha_1, \alpha_2$ be 2 paths connecting $u$ and $v$. The deletion of their common edges (if any) leads to a collection of edge-disjoint cycles. Each of these cycles consists of a subpath of $\alpha_1$ and a subpath of $\alpha_2$. Positivity of such a cycle implies the two subpaths share the same sign. Collecting all such subpaths together with the common edges, we see that $\alpha_1, \alpha_2$ have the same sign.