It is direct to check that

$$A^{\sigma^\tau} = D(\tau)^{-1} A^{\sigma} D(\tau).$$

Therefore, $-\Delta^{\sigma^\tau} = I - D^{-1} A^{\sigma^\tau} = D(\tau)^{-1} (I - D^{-1} A^{\sigma}) D(\tau)$.

That is, $-\Delta^{\sigma^\tau}$ and $-\Delta^{\sigma}$ are similar and hence share the same spectrum. $\square$

(II.6). Signed Cheeger constant.

We hope to quantify how close a given signed graph $\Gamma = (G, \sigma)$ is to be balanced. Based on Harary's balance theorem, we introduced the following concept.

~~Let~~ For any two subsets V_1, V_2 of V, denote

$$E^+(V_1, V_2) = \sum_{u \in V_1} \sum_{\substack{v \in V_2 \\ v \sim u \\ \sigma_{uv} = +1}} 1$$

and

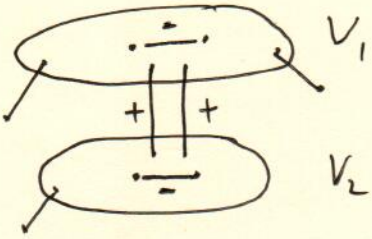
$$E^-(V_1, V_2) = \sum_{u \in V_1} \sum_{\substack{v \in V_2 \\ v \sim u \\ \sigma_{uv} = -1}} 1 .$$

When $V_1 = V_2$, we write $E^+(V_1)$, $E^-(V_1)$ for short.

Notice that, positive edges (negative edges) in $E^+(V_1)$ $(E^-(V))$ have been counted twice.

For any $V_1, V_2 \subseteq V$ s.t. $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 \neq \phi$, we define

$$\beta^{\sigma}(V_1, V_2) = \frac{2E^+(V_1, V_2) + E^-(V_1) + E^-(V_2) + |\partial(V_1 \cup V_2)|}{\mathrm{vol}(V_1 \cup V_2)}$$



The edges labeled in the figure left are the "obstacles" preventing the graph being balanced.

Definition. (Signed Cheeger constant, Atay - Liu. '14 ArXiv / '20 Discr. Math.)

The signed Cheeger constant for a signed graph $\Gamma = (G, \sigma)$. is defined as

$$h_1^\sigma = \min_{\substack{V_1, V_2 \subseteq V \\ V_1 \cap V_2 = \phi \\ V_1 \cup V_2 \neq \phi}} \beta^\sigma (V_1, V_2).$$

Remark: A direct observation is that

$$h_1^\sigma = 0 \iff \Gamma = (G, \sigma) \text{ has a balanced connected component.}$$

Proposition II.6.1. (Switching invariance). Let $\Gamma = (G, \sigma)$ be a signed graph. For any switching function $\tau : V \to \{+1, -1\}$. we have

$$h_1^{\sigma^\tau} = h_1^\sigma .$$

In fact, we have the following result.

Lemma II.6.1. For any switching fct $\tau : V \to \{+1, -1\}$, and any (V_1, V_2) s.t. $V_1, V_2 \subseteq V$, $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 \neq \phi$, there exists (V_1', V_2') s.t. $V_1' \cup V_2' = V_1 \cup V_2$, $V_1' \cap V_2' = \phi$, and

$$\beta^{\sigma^\tau} (V_1', V_2') = \beta^\sigma (V_1, V_2).$$

Proof of Prop. II.6.1: By the Lemma, we have $h_1^{\sigma^\tau} \le h_1^\sigma$.

Since $(\sigma^\tau)^\tau = \sigma$, we have $h_1^\sigma \leq h_1^{\sigma^\tau}$. $\square$

<u>Proof of Lemma II.6.1</u>:

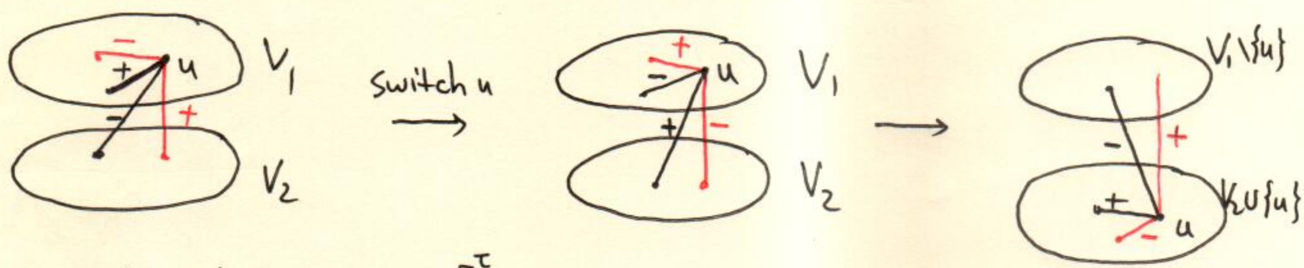
We only need to consider the case

$$\tau(x) = \begin{cases} -1 & x = u \\ +1 & \text{otherwise} \end{cases}$$

for some $u \in V$, that is, the case of a vertex switching.

If $u \notin V_1 \cup V_2$, we simply choose $V_1' = V_1$, $V_2' = V_2$.

If $u \in V_1 \cup V_2$, w.l.o.g., we assume $u \in V_1$, then we can choose $V_1' = V_1 \setminus \{u\}$, $V_2' = V_2 \cup \{u\}$.



$\beta^\sigma(V_1,V_2)$ counts red edges $\quad$ $\beta^{\sigma^\tau}(V_1,V_2)$ counts black edges $\quad$ $\beta^{\sigma^\tau}(V_1',V_2')$ counts red edges again.

By the above scheme, we observe that

$$\beta^\sigma(V_1,V_2) = \beta^{\sigma^\tau}(V_1',V_2') . \qquad \square$$

<u>Remark</u>: Observe that, we can reformulate the signed Cheeger constant as

$$h_1^\sigma = \min_{\substack{V_1,V_2 \subseteq V \\ V_1 \cap V_2 = \emptyset \\ V_1 \cup V_2 \neq \emptyset}} \frac{2E^+(V_1,V_2) + E^-(V_1) + E^-(V_2) + |\partial(V_1 \cup V_2)|}{\mathrm{vol}(V_1 \cup V_2)}$$

$$= \min_{\emptyset \neq \Omega \subseteq V} \; \min_{V_1 \sqcup V_2 = \Omega} \frac{2E^+(V_1,V_2) + E^-(V_1) + E^-(V_2) + |\partial\Omega|}{\mathrm{vol}(\Omega)}$$

$$= \min_{\emptyset \neq \Omega \subseteq V} \frac{\min_{V_1 \sqcup V_2 = \Omega} \left(2E^+(V_1,V_2) + E^-(V_1) + E^-(V_2)\right) + |\partial\Omega|}{\mathrm{vol}(\Omega)}$$

We have in fact shown the switching invariance of the quantities

$$\min_{V_1 \sqcup V_2 = \Omega} \left(2E^+(V_1,V_2) + E^-(V_1) + E^-(V_2)\right), \text{ for each } \Omega \subseteq V \quad \square$$

Observe that there is a one-to-one correspondence between the partitions of Ω and the switchings on Ω.

<u>Lemma II . 6.2</u> Let $\Gamma = (G, \sigma)$, where $G = (V, E)$, be a signed graph. Let $\Omega \subseteq V$ be any nonempty subset. Then we have

$$\min_{V_1 \sqcup V_2 = \Omega} \left(2E^+(V_1,V_2) + E^-(V_1) + E^-(V_2)\right) = 2 \cdot \min_{\tau: \Omega \to \{+1,-1\}} \sum_{\substack{\{u,v\} \in E \\ u,v \in \Omega}} |1 - \tau(u)\sigma_{uv}\tau(v)|$$

<u>Proof</u>: Every partition $V_1 \sqcup V_2 = \Omega$ corresponds to a switching function.

$$\tau_{V_1}(x) = \begin{cases} -1, & x \in V_1 \\ +1 & x \in V_2 \end{cases} \quad \text{or} \quad \tau^{o}_{V_2}(x) = \begin{cases} +1, & x \in V_1 \\ -1, & x \in V_2. \end{cases}$$

and

$$2E^+(V_1,V_2) + E^-(V_1) + E^-(V_2) = \frac{1}{2} \sum_{u \in \Omega} \sum_{\substack{v \in \Omega \\ v \sim u}} |1 - \tau_{V_i}(u)\sigma_{uv}\tau_{V_i}(v)|$$

$$= \sum_{\substack{\{u,v\} \in E \\ u,v \in \Omega}} |1 - \tau_{V_i}(u)\sigma_{uv}\tau_{V_i}(v)|.$$

Then the Lemma follows immediately. $\square$

<u>Remark</u>: Give τ, $\sum_{\substack{\{u,v\} \in E \\ u,v \in \Omega}} |1 - \tau(u)\sigma_{uv}\tau(v)|$ is (two(a)) counting the negative edges in Ω with the sign σ^τ. So, it can also be written as

$$\min_{\tau:\Omega\to\{+1,-1\}} \sum_{\substack{\{u,v\}\in E\\ u,v\in\Omega}} |1-\tau(u)\sigma_{uv}\tau(v)| = \min_{\sigma'\in[\sigma]} \sum_{\substack{\{u,v\}\in E\\ u,v\in\Omega}} |1-\sigma'_{uv}|$$

$$= \min_{\sigma'\in[\sigma]} 2\times \#\text{ negative edges in } \Omega \text{ w.r.t. } \sigma' . \quad \square$$

The <u>frustration index</u> ~~$e_{min}(\Gamma,\sigma)$~~ $e_{min}(G,\sigma)$, originally called the line index of balance by Frank Harary, is defined to be the minimal number of edges that is needed to be removed from the graph inorder to make it balance.

<u>Lemma II.6.3</u>: Let $\Gamma=(G,\sigma)$ be a signed graph, and Ω be a non-empty vertex subset. Denote the induced subgraph by $G_\Omega=(\Omega, E_\Omega)$ with the restricted sign σ_Ω. Then

$$\min_{\tau:\Omega\to\{+1,-1\}} \sum_{\{u,v\}\in E_\Omega} |1-\tau(u)\sigma_{uv}\tau(v)| = 2\, e_{min}(G_\Omega,\sigma_\Omega)$$

<u>Proof</u>: Let $\tau_0:\Omega\to\{+1,-1\}$ achieves the minimum in the LHS.

Let $E_0 := \left\{ \{u,v\}\in E_\Omega,\ \sigma^{\tau_0}_{uv} = \tau_0(u)\sigma_{uv}\tau_0(v) = -1 \right\}$. Then

$$LHS = 2|E_0|. \qquad (*)$$

Removing E_0 from E_Ω leaves the remaining part of (G_Ω,σ_Ω) balanced. Hence $|E_0|\geq e_{min}(G_\Omega,\sigma_\Omega)$.

If $|E_0| > e_{min}(G_\Omega,\sigma_\Omega)$, we can find another $E_1\subset E_\Omega$ with $|E_1|<|E_0|$, and $(\Omega, E_\Omega\setminus E_1)$ with the restricted sign is balanced.

Let $\tau_1:\Omega\to\{+1,-1\}$ be the switching function s.t.

$$\tau_1(u)\sigma_{uv}\tau_1(v) = +1 \quad \text{for any } \{u,v\}\in E_\Omega\setminus E_1.$$

Hence $\sum_{\{u,v\}\in E_\Omega} |1-\tau_1(u)\sigma_{uv}\tau_1(v)| \leq 2|E_1| < 2|E_0|$, contradicting $(*)$. $\square$

Given a signed graph $\Gamma = (G, \sigma)$, its frustration index can be calculated via its spanning trees.

<u>Lemma II.6.4</u>: (Harary & Kabell 1980, L.-Münch-Peyerimhoff 2019 Thm 4.10)

Let $\Gamma = (G,\sigma)$ be a connected signed graph. Then

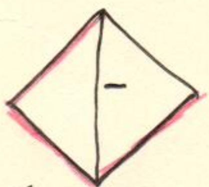
$$2e_{min}(G,\sigma) = \min_{T\in \mathbb{T}_G} \sum_{\{u,v\}\in E} \left|1 - \tau_T(u)\sigma_{uv}\tau_T(v)\right| \quad (\ast)$$

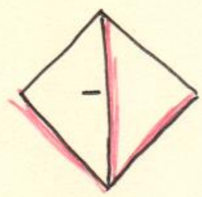
where $\mathbb{T}_G$ is the set of all spanning trees of G, $\tau_T : V \to \{+1,-1\}$ is a representative of the set

$$C_T(\Gamma) = \{\tau : V \to \{+1,-1\} : \tau_T(u)\sigma_{uv}\tau_T(v) = 1\}.$$

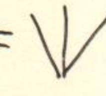
<u>Remark</u>: (i) $C_T(\Gamma)$ contains only two elements. If $\tau \in C_T(\Gamma)$, so does its negation. It does not matter whether we use τ or $-\tau \in C_T(\Gamma)$ for given T in $(\ast)$.

(ii). We do need taking the minimization over all ~~sign~~ spanning trees in $(\ast)$, since different choices of spanning trees can leads to different values. For example,





$T =$  $\to 2$

$T =$  $\to 4$.

<u>Proof</u>: By <u>Lemma II.6.3</u>, we only need show the fact $\tau_0 : V \to \{+1,-1\}$ achives the minimum of

$$\min_{\tau: V\to\{+1,-1\}} \sum_{\{u,v\}\in E} |1 - \tau(u)\sigma_{uv}\tau(v)|$$

has the following property:

$G_0 = (V, E_0)$, where $E_0 = \{ \{u,v\} \in E : \tau_0(u)\sigma_{uv}\tau_0(v) = 1 \}$, is connected. Then there is a spanning tree T_0 of G_0 s.t. $\tau_0 \in C_{T_0}(\Gamma)$.

Suppose $G_0 = (V, E_0)$ is not connected. Then $\exists$ a connected component $W \subsetneqq V$. We have $\underset{\{\{u,v\}\in E : u \in W, v \notin W\}}{\partial W \overset{\|}{\neq} \phi}$ since G is connected.

We observe that $\tau_0(u)\sigma_{uv}\tau_0(v) = -1$ for any $\{u,v\} \in \partial W$. Since, otherwise, $\{u,v\} \in E_0$, contradicting to the fact W is a connected component of $G_0 = (V_0, E)$.

We define $\tau_1 : V \to \{+1, -1\}$ via

$$\tau_1(x) = \begin{cases} -\tau_0(x) & \text{if } x \in W \\ \tau_0(x). & \text{if } x \in V \setminus W \end{cases}$$

Then $\displaystyle\sum_{\{u,v\}\in E} |1 - \tau_1(u)\sigma_{uv}\tau_1(v)|$

$$= \sum_{\substack{\{u,v\}\in E \\ u,v \in W}} |1 - \tau_1(u)\sigma_{uv}\tau_1(v)| + \sum_{\substack{\{u,v\}\in E \\ u,v \in V\setminus W}} |1 - \tau_1(u)\sigma_{uv}\tau_1(v)|$$

$$+ \sum_{\substack{\{u,v\}\in E \\ u\in W, v \in V\setminus W}} |1 - \tau_1(u)\sigma_{uv}\tau_1(v)|$$

$$= \sum_{\substack{\{u,v\}\in E \\ \text{either } u,v \in W \\ \text{or } u,v \notin W}} |1 - \tau_0(u)\sigma_{uv}\tau_0(v)| + \sum_{\substack{\{u,v\}\in E \\ u\in W, v \notin W}} |1 + \underbrace{\tau_0(u)\sigma_{uv}\tau_0(v)}_{=-1}|$$

since $\partial W \neq \phi$

$$< \sum_{\{u,v\}\in E} |1 - \tau_0(u)\sigma_{uv}\tau_0(v)|,$$ contradicting to the choice of τ_0. $\square$

In conclusion, we have several expression for the signed Cheeger constant.

$$h_1^\sigma = \min_{\phi \neq \Omega \subseteq V} \frac{\min\limits_{V_1 \sqcup V_2 = \Omega} \left(2E^+(V_1,V_2) + E^-(V_1) + E^-(V_2)\right) + |\partial\Omega|}{\mathrm{vol}(\Omega)}$$

$$= \min_{\phi \neq \Omega \subseteq V} \frac{2e_{\min}(G_\Omega, \sigma_\Omega) + |\partial\Omega|}{\mathrm{vol}(\Omega)}.$$

~~In particular, $\min_{V_1 \sqcup V_2 = \Omega} \beta^\sigma(V_1,V_2)$, where $\beta^\sigma(V_1,V_2)$ can be~~

The quantity $\dfrac{2e_{\min}(G_\Omega, \sigma_\Omega) + |\partial\Omega|}{\mathrm{vol}(\Omega)}$ can be considered as a signed expansion of Ω.

<u>Remark</u>. The signed Cheeger constant ~~unifies~~ covers ~~a~~ various constants introduced by Trevisan, Bauer and Jost, it is also closely related to ~~some~~ a constant~~s~~ of Desai and Rao, and its signed version by Y.-P. Hou. (侯耀平教授，湖南师范大学).

(II.7) Signed Cheeger inequality.

<u>Theorem II.7.1</u> (Atay-L.) Given a signed graph, we have

$$\frac{(h_1^\sigma)^2}{2} \leq \lambda_1^\sigma \leq 2h_1^\sigma.$$