In conclusion, we have several expressions for the signed Cheeger constant.

$$h_1^\sigma = \min_{\phi\neq\Omega\subseteq V} \frac{\min\limits_{V_1\sqcup V_2=\Omega}\left(2E^+(V_1,V_2)+E^-(V_1)+E^-(V_2)\right)+|\partial\Omega|}{\mathrm{vol}(\Omega)}$$

$$= \min_{\phi\neq\Omega\subseteq V} \frac{2e_{\min}(G_\Omega,\sigma_\Omega)+|\partial\Omega|}{\mathrm{vol}(\Omega)} = \min_{\phi\neq\Omega\subseteq V} \frac{\min\limits_{\tau:\Omega\to\{\pm1\}} \sum\limits_{\{u,v\}\in E_\Omega} |1-\tau(u)\sigma_{uv}\tau(v)|+|\partial\Omega|}{\mathrm{vol}(\Omega)}.$$

~~In particular, $\min\limits_{V_1\sqcup V_2=\Omega} \beta^\sigma(V_1,V_2)$, where $\beta^\sigma(V_1,V_2)$ can be~~

The quantity $\dfrac{2e_{\min}(G_\Omega,\sigma_\Omega)+|\partial\Omega|}{\mathrm{vol}(\Omega)}$ can be considered as a signed expansion of Ω.

<u>Remark</u>. The signed Cheeger constant ~~unifies~~ covers various constants introduced by Trevisan, Bauer and Jost, it is also closely related to ~~some~~ a constant of Desai and Rao, and its signed version by Y.-P. Hou. (侯耀平教授, 湖南师范大学).

(II.7) Signed Cheeger inequality.

<u>Theorem II.7.1</u> (Atay-L.) Given a signed graph, we have

$$\frac{(h_1^\sigma)^2}{2} \leq \lambda_1^\sigma \leq 2h_1^\sigma.$$

<u>Remark</u>: By switching invariance of the eigenvalues and Cheeger constants, we have

- If $\sigma\in[-]$, where $[-]$ is the switching class of all -1 signs, this ineq. is $\frac{\beta^2}{2}\leq 2-\lambda_N\leq 2\beta$, where λ_N is the maximal eigenvalue of the underlying graph G.

- If $\sigma \in [+]$, ($[+]$ is the switching class of all $+1$ signs), the inequality reduces to the trivial fact.

$$0 \leq \lambda_1 \leq 2\cdot 0 \quad \text{i.e. } \lambda_1 = 0.$$

where λ_1 is the smallest eigenvalue of the underlying graph G.

<u>Proof of $\lambda_1^\sigma \leq 2h_1^\sigma$</u>: We use the following expression of h_1^σ:

$$h_1^\sigma = \min_{\phi \neq \Omega \subseteq V} \frac{\min_{\tau: \Omega \to \{+1,-1\}} \sum_{\{u,v\} \in E_\Omega} |1 - \tau(u)\sigma_{uv}\tau(v)| + |\partial\Omega|}{vol(\Omega)}$$

Let $\Omega_0 \subseteq V$ be the subset that achieves h_1^σ above, and $\tau_0: \Omega_0 \to \{+1,-1\}$ the switching function which achieves $\min_{\tau: \Omega \to \{+1,-1\}} \sum_{\{u,v\} \in E_\Omega} |1 - \tau(u)\sigma_{uv}\tau(v)|$.

Define the function $f_0: V \to \mathbb{R}$ s.t.

$$f_0(x) := \begin{cases} \tau_0(x), & \text{if } x \in \Omega_0 \\ 0, & \text{otherwise.} \end{cases}$$

Then
$$\lambda_1^\sigma \leq R^\sigma(f_0) = \frac{\sum_{\{u,v\} \in E} (f(u) - \sigma_{uv} f(v))^2}{\sum_{u \in V} f(u)^2 d_u}$$

$$= \frac{\sum_{\{u,v\} \in E_{\Omega_0}} |\tau_0(u) - \sigma_{uv}\tau_0(v)|^2 + |\partial\Omega_0|}{vol(\Omega_0)}$$

$$= \frac{\sum_{\{u,v\} \in E_{\Omega_0}} |1 - \tau_0(u)\sigma_{uv}\tau_0(v)|^2 + |\partial\Omega_0|}{vol(\Omega_0)}$$

$$\leq \frac{2\sum_{\{u,v\} \in E_{\Omega_0}} |1 - \tau_0(u)\sigma_{uv}\tau_0(v)| + |\partial\Omega_0|}{vol(\Omega_0)} \leq 2h_1^\sigma. \quad \square$$

Proof of $\lambda_1^\sigma \geq \frac{(h_1^\sigma)^2}{2}$:

We consider the indicator function: For any function $f: V \to \mathbb{R}$, consider

$$\Omega_f(t) = \{x \in V : f(x)^2 > t\}, \quad t \in [0, \infty)$$

We define an indicator function $\chi_f^t : V \to \mathbb{R}$ s.t.

$$\chi_f^t(x) = 0 \text{ if } x \notin \Omega_f(t) \text{ and}$$
$$|\chi_f^t(x)| = 1 \text{ if } x \in \Omega_f(t)$$

More specifically, we divide $\Omega_f(t)$ into two subsets

$$V_f(t) := \{x \in V : f(x) > \sqrt{t}\},$$

and $V_f(-t) := \{x \in V : f(x) < -\sqrt{t}\}$,

and the indicator function is

$$\chi_f^t(x) = \begin{cases} 1, & \text{if } x \in V_f(t) \\ -1, & \text{if } x \in V_f(-t) \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{x \in V} |\chi_f^t(x)| d_x = vol(\Omega_f(t)), \text{ and}$$

$$\sum_{\{x,y\} \in E} |\chi_f^t(x) - \sigma_{xy}\chi_f^t(y)| = \sum_{\{x,y\} \in E_{\Omega_f(t)}} |\chi_f^t(x) - \sigma_{xy}\chi_f^t(y)| + |\partial \Omega_f(t)|$$

Notice that $\chi_f^t : \Omega_f(t) \to \{+1, -1\}$.

We will study the behavior of the ratio

$$I_f := \int_0^\infty \sum_{\{x,y\} \in E} |\chi_f^t(x) - \sigma_{xy}\chi_f^t(y)| dt \Big/ \int_0^\infty \sum_{x \in V} |\chi_f^t(x)| d_x dt.$$

On the one hand, there exists $t_0 \in [0,\infty)$ s.t.

$$\frac{\min\limits_{\tau:\Omega_f(t_0)\to\{+1,-1\}} \sum\limits_{\{x,y\}\in E_{\Omega_f(t_0)}} |1-\tau(x)\sigma_{xy}\tau(y)| + |\partial\Omega_f(t_0)|}{vol(\Omega_f(t_0))} \leq I_f. \quad (\star)$$

On the other hand, we have:

<u>Lemma</u>: $I_f \leq \sqrt{2R^\sigma(f)}$.

<u>Proof</u>: First, $\int_0^\infty \sum_{x\in V} |\chi_f^t(x)|\, d_x\, dt = \sum_{x\in V}\int_0^{f^2(x)} dt\; d_x$

$$= \sum_{x\in V} f(x)^2 d_x. \quad ①$$

Secondly, we have

$$\int_0^\infty \sum_{\{x,y\}\in E} |\chi_f^t(x) - \sigma_{xy}\chi_f^t(y)|\, dt$$

$$= \sum_{\{x,y\}\in E} \int_0^\infty |\chi_f^t(x) - \sigma_{xy}\chi_f^t(y)|\, dt. \quad ②$$

<u>Claim</u>: For $\{x,y\}\in E$ $\int_0^\infty |\chi_f^t(x) - \sigma_{xy}\chi_f^t(y)|\, dt \leq |f(x) - \sigma_{xy} f(y)|\,(|f(x)|+|f(y)|)$.

<u>Proof</u>: This is an extension of Lemma II.3.4 (page 85).
Observe that, this is a property that only relates to the value of f at x and y. If we consider a new function g s.t. $g(x) = f(x)$, $g(y) = -\sigma_{xy} f(y)$.

Then we have $\chi_f^t(x) = \chi_g^t(x)$, $-\sigma_{xy}\chi_f^t(y) = \chi_g^t(y)$

Hence by applying Lemma II.3.4, we have

$$\int_0^\infty |\chi_f^t(x) - \sigma_{xy}\chi_f^t(y)|\, dt = \int_0^\infty |\chi_g^t(x) + \chi_g^t(y)|\, dt \overset{\text{Lemma II.3.4.}}{\leq} |g(x)+g(y)|\,(|g(x)|+|g(y)|)$$

$$= |f(x) - \sigma_{xy} f(y)| \left(|f(x)| + |f(y)|\right). \qquad \square.$$

By the claim, we have

$$\sum_{\{x,y\}\in E} \int_0^\infty |\chi^t_{f(x)} - \sigma_{xy} \chi^t_f(y)| \, dt$$

$$\leq \sum_{\{x,y\}\in E} |f(x) - \sigma_{xy} f(y)| \left(|f(x)| + |f(y)|\right)$$

$$\leq \left(\sum_{\{x,y\}\in E} |f(x) - \sigma_{xy} f(y)|^2\right)^{1/2} \underbrace{\left(\sum_{\{x,y\}\in E} \left(|f(x)| + |f(y)|\right)^2\right)^{1/2}}_{\leq \left(\sum_{\{x,y\}\in E}\left(2|f(x)|^2 + 2|f(y)|^2\right)\right)^{1/2}}$$

$$= \left(2 \sum_x |f(x)|^2 d_x\right)^{1/2} \qquad ③$$

Combining ①, ② and ③, we arrive at

$$I_f \leq \sqrt{2 R^\sigma(f)}. \qquad \square$$

Combining (✱) and Lemma leads to

$$h_1^\sigma \leq I_f \leq \sqrt{2R^\sigma(f)}.$$

Setting f to be the eigenfunction to λ_1^σ provides

$$h_1^\sigma \leq \sqrt{2\lambda_1^\sigma}. \qquad \square.$$

(II.8) Higher order Cheeger inequalities

For convenience, we denote for $\phi \neq \Omega \subseteq V$.

$$\phi^\sigma(\Omega) := \frac{e_{\min}(G_\Omega, \sigma_\Omega) + |\partial\Omega|}{vol(\Omega)}.$$

$$= \frac{\min_{\tau:\Omega\to\{+1,-1\}} \sum_{\{x,y\}\in E} |1 - \tau(x)\sigma_{xy}\tau(y)| + |\partial\Omega|}{vol(\Omega)}.$$

We can introduce a family of multi-way Cheeger constants as below.

Definition. Given $1 \le k \le N$, the k-way Cheeger constant h_k^σ of a signed graph $\Gamma = (G, \sigma)$ is defined as

$$h_k^\sigma := \min_{\substack{\Omega_1, \dots, \Omega_k \subseteq V \\ \Omega_i \neq \phi, \forall i \\ \Omega_i \cap \Omega_j = \phi, \forall i \neq j}} \max_{1 \le i \le k} \phi^\sigma(\Omega_i).$$

The subsets $\{\Omega_i\}_{i=1}^k$ satisfying $\Omega_i \neq \phi, \forall i$, $\Omega_i \cap \Omega_j = \phi$ $\forall i \neq j$ is called a k-subpartition of V.

Remark (1) Given $\Gamma = (G, \sigma)$, then

$h_k^\sigma = 0 \Leftrightarrow \Gamma$ has at least k balanced conn. components.

(2) By definition, we have the monotonicity $h_k^\sigma \le h_{k+1}^\sigma$.

(3) When $k=1$, this is the Cheeger constant we studied in (II.6) and (II.7).

When $k=2$, and $\sigma \in [+]$, ~~we see~~

$$h_2^\sigma = h_2^+ = \min_{\substack{\Omega_1, \Omega_2 \subseteq V \\ \Omega_i \neq \phi \\ \Omega_1 \cap \Omega_2 = \phi}} \max_{i=1,2} \frac{|\partial \Omega_i|}{\mathrm{vol}(\Omega_i)}$$

reduces to the classical Cheeger constant.

Theorem: Let $\Gamma = (G, \sigma)$ be a signed graph. Then we have

$$\lambda_k^\sigma \le 2 h_k^\sigma \quad \text{for any } 1 \le k \le N.$$

Proof: Recall from the minimax principle that

$\lambda_k^\sigma = \min\limits_{L \in P^k} \max \left\{ R^\sigma(f),\ f \in L \setminus \{0\} \right\}$, where P^k is the collection of all k-dim. linear subspace of $\ell^2(V)$.

~~Consider the families of functions $f_1, \cdots, f_k$ such that~~

Suppose $\{\Omega_i\}_{i=1}^k$ is the k-subpartition of V that achieves the k-way Cheeger constant, and $\{\tau_i\}_{i=1}^k$ are the switching functions achieves the $2\times$ frustration index of each G_{Ω_i} (We can call τ_i the optimal switching function of Ω_i). Define

$$f_i(x) = \begin{cases} \tau_i(x), & \text{if } x \in \Omega_i \\ 0, & \text{otherwise.} \end{cases}$$

Then f_i's are disjointly supported, and hence orthogonal to each other. Therefore $\mathrm{span}\{f_1, \cdots, f_k\} \in P^k$. Then

$$\lambda_k^\sigma \le \max_{f \in \mathrm{span}\{f_1, \cdots, f_k\} \setminus \{0\}} R^\sigma(f)$$

$$= \max_{\substack{a_1, \cdots, a_k \in \mathbb{R} \\ a_1 \cdots a_k \neq 0}} R^\sigma\left(\sum_{i=1}^k a_i f_i\right),$$

where
$$R^\sigma\left(\sum_{i=1}^k a_i f_i\right) = \frac{\sum_{\{u,v\}} \left(\sum_{i=1}^k a_i f_i(u) - \sigma_{uv} \sum_{j=1}^k a_j f_j(v)\right)^2}{\sum_{u \in V} \left(\sum_{i=1}^k a_i f_i(u)\right)^2 d_u}.$$

$$\sum_{u \in V} \left(\sum_{i=1}^k a_i f_i(u)\right)^2 d_u = \sum_{u \in V} \sum_{i=1}^k a_i^2 f_i(u)^2 d_u = \sum_{i=1}^k a_i^2 \mathrm{vol}(\Omega_i). \quad (1)$$

$$\sum_{\{u,v\} \in E} \left(\sum_{i=1}^k \left(a_i f_i(u) - \sigma_{uv} a_i f_i(v)\right)\right)^2 \le \sum_{\{u,v\} \in E} 2 \sum_{i=1}^k \left(a_i f_i(u) - \sigma_{uv} a_i f_i(v)\right)^2$$

For given $\{u,v\} \in E$, ~~this is~~ here we have at most 2 nontrivial summands.

(2)

However, this estimate is too ~~generous~~. "generous"!

When $\{u,v\} \in E_{\Omega_{i_0}}$ for some i_0, the summation

$$\sum_{i=1}^{k} \left(a_i f_i(u) - \sigma_{uv} a_i f_i(v)\right)$$

has only one nonzero summand. Therefore, define

$$C_{\{u,v\}} = \begin{cases} 1 & \text{if } \{u,v\} \in \Omega_i \text{ for some } i \\ 2 & \text{otherwise.} \end{cases}$$

We have

$$\sum_{\{u,v\} \in E} \left(\sum_{i=1}^{k} a_i \left(f_i(u) - \sigma_{uv} f_i(v)\right) \right)^2$$

$$\leq \sum_{\{u,v\} \in E} C_{\{u,v\}} \sum_{i=1}^{k} a_i^2 \left(f_i(u) - \sigma_{uv} f_i(v)\right)^2$$

$$= \sum_{i=1}^{k} a_i^2 \left(\sum_{\{u,v\} \in E_{\Omega_i}} \left(f_i(u) - \sigma_{uv} f_i(v)\right)^2 + 2 \sum_{\{u,v\} \in \partial\Omega_i} \left(f_i(u) - \sigma_{uv} f_i(v)\right)^2 \right)$$

$$\leq \sum_{i=1}^{k} a_i^2 \left(2 \sum_{\{u,v\} \in E_{\Omega_i}} \left| f_i(u) - \sigma_{uv} f_i(v) \right| + 2 \left| \partial\Omega_i \right| \right)$$

$$\leq 2 \sum_{i=1}^{k} a_i^2 \left(\sum_{\{u,v\} \in E_{\Omega_i}} \left| \tau_i(u) - \sigma_{uv} \tau_i(v) \right| + \left| \partial\Omega_i \right| \right)$$

$$= 2 \sum_{i=1}^{k} a_i^2 \left(2 e_{min}(G_{\Omega_i}, \sigma_{\Omega_i}) + \left| \partial\Omega_i \right| \right) \qquad (2)$$

Combining (1) & (2) yields

$$R^\sigma\left(\sum_{i=1}^{k} a_i f_i\right) \leq \frac{2 \sum_{i=1}^{k} a_i^2 \left(2 e_{min}(G_{\Omega_i}, \sigma_{\Omega_i}) + |\partial\Omega_i| \right)}{\sum_{i=1}^{k} a_i^2 \, \mathrm{vol}(\Omega_i)}$$

$$\leq 2 \max_{i=1,\dots,k} \phi^\sigma(\Omega_i).$$

That means $\lambda_k^\sigma \leq 2 h_k^\sigma$. □