We apply the following procedure to $\{\hat{S}_i\}_i$. If we can find two sets of the subpartition, say $\hat{S}_i$, $\hat{S}_j$, s.t.

$$\mu_F(\hat{S}_i) \leq \frac{1}{2k}\mu_F(V), \quad \mu_F(\hat{S}_j) \leq \frac{1}{2k}\mu_F(V),$$

then replace them by $\hat{S}_i \cup \hat{S}_j$. Thus, when we stop, we obtain the sets, $T_1, T_2, \cdots, T_l$ for some l, s.t.

$$\mu_F(T_i) \leq \frac{1}{k(1-r^2)}\mu_F(V). \quad \forall i = 1, 2, \cdots, l.$$

and $\mu_F(T_i) \geq \frac{1}{2k}\mu_F(V)$, $\forall i \in 1, \cdots, l-1$.

Setting $r = \frac{1}{3\sqrt{k}}$ and $\delta = \frac{1}{4k}$, we check.

$$(k-1)\frac{1}{k(1-r^2)} < 1 - \delta - \frac{1}{2k}.$$

This implies that $l \geq k$. Moreover, if we ~~restate~~ redefine $T_k = \cup_{j=k}^{l} T_k$, we have

$$\mu(T_k) \geq \frac{1}{2k}\mu_F(V).$$

The subpartition $\{T_i\}_{[k]}$ is what we want. $\square$.

Random partition theory.

It remains to show Theorem 4. (p. 122) in order to complete the whole ~~graph~~ proof.

① Metric doubling property.

Let (X, d) be a metric space. The metric doubling constant ρ_X of (X, d) is defined as

$$\rho_X = \inf\left\{ c \in \mathbb{N} : \forall x \in X,\ r>0,\ \exists x_1, \dots, x_c \in X \text{ s.t. } B(x,r) \subseteq \bigcup_{i=1}^{c} B(x_i, \tfrac{r}{2}) \right\}.$$

If $\rho_X < \infty$, (X,d) is called a metric doubling space.

ρ_X is the smallest value s.t. every ball in X can be covered by ρ_X balls of half the radius.

Recall an ε-net of (X,d) is a subset $Y \subseteq X$ s.t.
$X \subset \bigcup_{y\in Y} B(y,\varepsilon)$ and $d(y,y') > \varepsilon$ for any $y,y' \in Y$.

<u>Proposition 1</u>: Let (X,d) be a metric doubling space with constant ρ_X. ~~Let Y be an r-net of X.~~ If all pairwise distances in $Y \subseteq X$ are $\geq r$. (e.g., Y is an r-net). Then for any $x \in X$, and any radius $t \geq r$, we have

$$|B(x,t) \cap Y| \leq \rho_X^{\lceil \log_2 \frac{2t}{r} \rceil}.$$

<u>Proof</u>: let $n \in \mathbb{N}$ be such that $\frac{t}{(\frac{r}{2})} = 2^n$.
By definition of an r-net, we have

$$B(y,\tfrac{r}{2}) \cap B(y',\tfrac{r}{2}) = \phi, \quad \forall\, y,y' \in Y,\ y \neq y'.$$

Therefore

$$|B(x,t) \cap Y| \leq (\rho_X)^n \leq \rho_X^{\lceil \log_2 \frac{2t}{r} \rceil}. \qquad \square$$

<u>Definition</u>: The metric doubling dimension of (X,d), $\dim_d(X) := \log_2 \rho_X$.

② <u>Padded random partitions</u>.

<u>Thm 5</u>: Let (X,d) be a finite metric subspace of ~~(Z,d)~~ (M,d).
Then for every $r>0$, $\delta \in (0,1)$, there exists a random

partition. $\mathcal{P}$ of (X,d), i.e., a distribution ν over all possible partitions of (X,d), such that

(i) $\operatorname{diam}(S) \le r$, for any S in every partition in the support of ν.

(ii) $\mathbb{P}\left[B_{\frac{r}{\alpha}}(x) \subseteq \mathcal{P}(x)\right] \ge 1-\delta$ for all $x \in X$, where $\alpha = \frac{32\dim_d(M)}{\delta}$.

<u>Proof</u>: We first construct such a random partition and then show the partition has the required property.

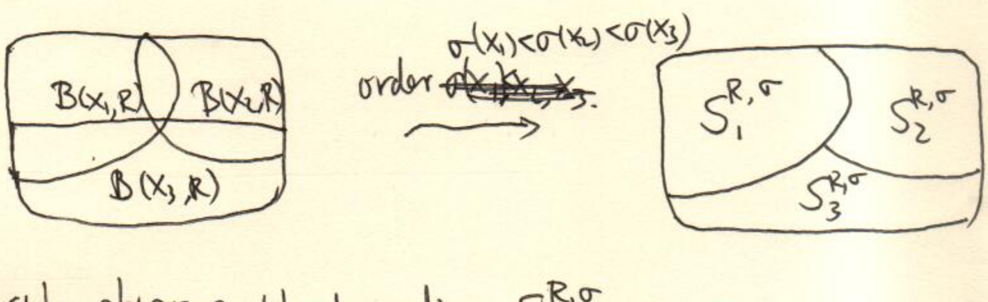
Take a $\frac{r}{4}$-net of (X,d). Denote it by $\{x_i\}_{i=1}^m$. By def, we have

$$X = \bigcup_{i=1}^m B(x_i, \tfrac{r}{4}), \quad \text{and} \quad d(x_i, x_j) \ge \frac{r}{4}, \text{ for any } i \ne j.$$

For any given $R \in \left(\frac{r}{4}, \frac{r}{2}\right]$, we construct a partition of X as follows: A permutation σ of the set $\{1, \cdots, m\}$ provides an order for all points $\{x_i\}_{i=1}^m$. This order is used to define for each $i = 1, \cdots, m$.

$$S_i^{R,\sigma} := \{x \in X \mid x \in B(x_i, R) \text{ and } \sigma(i) < \sigma(j) \text{ for all } j \in \{1, \cdots, m\} \text{ with } x \in B(x_j, R)\}.$$

This is actually quite intuitive: First come, first go.



First observe that $\operatorname{diam} S_i^{R,\sigma} \le 2R \le r$. as designed.

Then $\mathcal{P}^{R,\sigma} := \{S_i^{R,\sigma}\}_{i=1}^m$ is a partition of X.

Now let σ be a uniformly random permutation of $\{1,\cdots,m\}$ and R is chosen uniformly at random from the interval $(\frac{r}{4},\frac{r}{2}]$.

Next we show such a random partition satisfies (ii).

Fix a point $x\in X$, and some $t\in[0,\frac{r}{4}]$. We consider the opposite event to $B(x,t)\subseteq P(x)$. That is, $\exists\ S_i^{R,\sigma}$ s.t. $S_i^{R,\sigma}\cap B(x,t)\neq\phi$ but $B(x,t)\nsubseteq S_i^{R,\sigma}$.

~~That i~~ We say $B(x,t)$ is cut by $S_i^{R,\sigma}$. We will estimate the probability that $B(x,t)$ is cut.

(i) For a given $B(x,t)$, those x_i s.t. $S_i^{R,\sigma}$ cut $B(x,t)$ must satisfies

$$d(x_i,x)<R+t\leq\frac{r}{2}+t.$$

Let us denote by $W:=B(x,\frac{r}{2}+t)\cap\{x_i\}_{i=1}^m$.

Then by Proposition 1, we estimate

$$\ell:=|W|\leq\rho_Y^{\lceil\log_2\frac{r+2t}{r/4}\rceil}=\rho_Y^{\lceil\log_2(4+\frac{8t}{r})\rceil}\leq\rho_Y^3$$

$$=\left(2^{\dim_d Y}\right)^3=8^{\dim_d Y}.$$

Let us arrange the points $\omega_1,\omega_2,\cdots,\omega_\ell\in W$ in order of increasing distance from x.

(ii) A necessary condition for $S_{\omega_j}^{R,\sigma}$ cut $B(x,t)$ is that $R\in[d(x,\omega_j)-t\ ,\ d(x,\omega_j)+t]:=I_j$.

Finally, we write $\mathcal{E}_j$ for the event that ω_j is the minimal element in W (according to σ) for which $S^{R,\sigma}_{\omega_j}$ cut $B(x,t)$. (129)

Then

$$\mathbb{P}[B(x,t) \text{ is cut}] \le \sum_{j=1}^{\ell} \mathbb{P}(\mathcal{E}_j)$$

$$= \sum_{j=1}^{\ell} \mathbb{P}[R \in I_j] \cdot \mathbb{P}(\mathcal{E}_j \mid R \in I_j)$$

$$\le \sum_{j=1}^{\ell} \frac{2t}{\frac{r}{4}} \cdot \frac{1}{j}$$

we need ω_j is ordered before $\omega_1, \dots, \omega_{j-1}$ since if $S^{R,\sigma}_{\omega_j}$ cut $B(x,t)$, and if ω_i, $i \le j-1$ is ordered before ω_j by σ, then $S^{R,\sigma}_{\omega_i}$ must also cut $B(x,t)$. Then ω_j is not the minimal element in W for which $S^{R,\sigma}_{\omega_i}$ cut $B(x,t)$ any more.

Using $\sum_{j=1}^{\ell} \frac{1}{j} \le 1 + \int_1^{\ell} \frac{1}{x}\,dx = 1 + \ln \ell$

we have
$$\mathbb{P}[B(x,t) \text{ is cut}] \le \frac{8t}{r}(1 + \ln \ell)$$

$$\le \frac{8t}{r}(1 + \ln 8 \cdot \dim_d Y).$$

$$\le \frac{8t}{r}(1 + \ln 8) \dim_d Y$$

$$\le \frac{32t}{r} \dim_d Y.$$

Therefore, $\mathbb{P}[B(x,t) \subseteq P(x)] \ge 1 - \frac{32t}{r} \dim_d Y$.

Set $t = \frac{\delta r}{32 \dim_d Y}$ ~~that~~ fulfills our purpose.

That is set $\alpha = \frac{32 \dim_d Y}{\delta}$.

$\square$.