③ Measure doubling implies Metric doubling.

A Borel measure μ on (X,d) is called a doubling measure if there exists a number C_μ s.t. $\forall x \in X$, $r>0$.

$$0 < \mu(B(x,r)) \le C_\mu \, \mu(B(x,\tfrac{r}{2})) < +\infty.$$

Similarly, we call $\dim_\mu(X) = \log_2 C_\mu$ the measure doubling dimension.

Lemma. Let (X,d) be a metric space with a doubling measure μ. then $\dim_d X \le 4\dim_\mu(X)$.

Proof. Let us take any $x\in X$, and a ball $B(x,r)$ with radius r. Take a $\frac{r}{2}$-net $\{x_i\}_{i=1}^{c}$ of $B(x,r)$. Then all balls $\{B(x_i,\frac{r}{4})\}_{i=1}^{c}$ are mutually disjoint. Moreover, by triangular inequality, we have

$$\bigcup_{i=1}^{c} B(x_i,\tfrac{r}{4}) \subseteq B(x,\tfrac{5r}{4}).$$

Hence $$\sum_{i=1}^{c}\mu(B(x_i,\tfrac{r}{4})) \le \mu(B(x,\tfrac{5r}{4})) \qquad (*)$$

For any x_{i_0} from the $\frac{r}{2}$-net, we have via triangle ineq.

$$B(x,\tfrac{5r}{4}) \subseteq B(x_{i_0}, r+\tfrac{5}{4}r) \qquad \left(\begin{array}{l}\forall y\in B(x,\tfrac{5r}{4})\\ d(x_{i_0},y)\le d(x,x_{i_0}) \\ \quad +d(x,y) \\ \le r+\tfrac{5r}{4}\end{array}\right)$$

Therefore we continue $(*)$ to get

$$\sum_{i=1}^{c}\mu(B(x_i,\tfrac{r}{4})) \le \mu(B(x,\tfrac{5r}{4})) \le \mu(B(x_{i_0},\tfrac{9}{4}r))$$

$$\underset{\text{measure doubling}}{\le} C_\mu^{\lceil \log_2 9\rceil}\mu(B(x_{i_0},\tfrac{r}{4})) = C_\mu^{4}\,\mu(B(x_{i_0},\tfrac{r}{4}))$$

First, c is finite. Otherwise, ~~suppose c =~~

$$0 < \mu(B(x_1,\tfrac{r}{4})) \leq \sum_{i=1}^{\infty} \mu(B(x_i,\tfrac{r}{4})) \leq \underbrace{C_\mu^4\, \mu(B(x_{i_0},\tfrac{r}{4}))}_{<+\infty} \quad \forall i_0$$

Therefore, $\sum_{i=1}^{\infty} \mu(B(x_i,\frac{r}{4}))$ converges. This implies $\lim_{i\to\infty} \mu(B(x_i,\frac{r}{4})) = 0$

This contradicts to $0 < \mu(B(x_1,\frac{r}{4})) \leq C_\mu^4\, \mu(B(x_{i_0},\frac{r}{4}))$, $\forall$ all i_0.

Secondly, we can choose i_0 s.t. $\mu(B(x_{i_0},\frac{r}{4})) = \min_i \mu(B(x_i,\frac{r}{4}))$

Then we have

$$c\cdot \mu(B(x_{i_0},\tfrac{r}{4})) \leq \sum_{i=1}^{c} \mu(B(x_{i_0},\tfrac{r}{4})) \leq C_\mu^4\, \mu(B(x_{i_0},\tfrac{r}{4}))$$

$$\Rightarrow \quad c \leq C_\mu^4 \quad \Rightarrow \quad \log_2 c \leq 4 \log_2 C_\mu .$$

That is, $\dim_d X \leq 4 \dim_\mu(X)$. $\square$

<u>Rmk</u>: For our purpose, we need apply Lemma to the metric space $(P^{k-1}\mathbb{R}, \bar{d})$, where $\bar{d}([x],[y]) := \min\{\|x+y\|, \|x-y\|\}$

Direct to check that for the Rie. distance d_{Rie}, we have $\quad c\, d_{Rie} \leq \bar{d} \leq d_{Rie}$

Then
$$\frac{\mu_{Rie}(B_{\bar{d}}(x,r))}{\mu_{Rie}(B_{\bar{d}}(x,\frac{r}{2}))} \leq \frac{\mu_{Rie}(B_{d_{Rie}}(x,\frac{r}{c}))}{\mu_{Rie}(B_{d_{Rie}}(x,\frac{r}{2}))} \leq \left(\frac{2}{c}\right)^{k-1}$$

$$\Rightarrow \quad \dim_{\bar{d}}(P^{k-1}\mathbb{R}) \leq 4 \log_2 \left(\frac{2}{c}\right)^{k-1} = \underbrace{4\left(\log_2 \frac{2}{c}\right)\cdot(k-1)}_{\approx\, C\cdot k} . \quad \square$$

Then, we finish the proof of Higher order Cheeger ineq.

Let's look again at the Higher order Cheeger inequality: For any $(G=(V,E),\sigma)=\Gamma$, any $1\le k\le |V|$, there exists an absolute constant s.t.

$$\frac{C}{k^6}(h_k^\sigma)^2 \le \lambda_k^\sigma \le 2h_k^\sigma .$$

Apparently, there is an "unbalance" in the order of the Cheeger constant in the lower and upper bounds. This leaves a lot more to explore about Cheeger inequalities.

(II.9) Improved Cheeger inequalities.

Our main result in this section is the following.

<u>Theorem II.9.1</u>: For any signed graph $\Gamma=(G,\sigma)$ and any $k\in\{1,2,\cdots,|V|\}$, ~~we have~~ there exists an absolute constant C s.t.

$$h_1^\sigma \le C k \frac{\lambda_1^\sigma}{\sqrt{\lambda_k^\sigma}} .$$

<u>Remark</u>: Of course, when $k=1$, the ineq. reduces to Cheeger inequality.

Our first step in proving Theorem II.9.1 is the following:

~~Lemma II.9.2~~: For any nontrivial $f: V\to\mathbb{R}$, show the existence of Ω in the support of f s.t.

$$\phi^\sigma(\Omega) \lesssim C\,R^\sigma(f) + \text{additional terms}.$$

We first show a ~~ϕ~~ reflection result.

<u>Lemma II.9.2</u> For any nontrivial function $f: V \to \mathbb{R}$. there exists $t_0 \in [0, \max_{u\in V} |f(u)|]$ s.t that

$$\phi^\sigma(\Omega_f(t_0)) \leq \frac{\sum_{\{u,v\}} |f(u) - \sigma_{uv} f(v)|}{\sum_u d_u |f(u)|}$$

where $\Omega_f(t_0) := \{u \in V : |f(u)| > t_0\}$

<u>Proof</u>: We can assume $\max_{u\in V} |f(u)| = 1$ w.l.o.g.

We define an indicator function for $t \in [0,1]$

$$\chi_f^t : V \to \mathbb{R} \quad \text{s.t.} \quad \chi_f^t(u) = \begin{cases} 1, & f(u) > t \\ -1, & f(u) < -t \\ 0, & \text{otherwise} \end{cases}$$

Consider the ratio:

$$\frac{\sum_{\{u,v\}} |\chi_f^t(u) - \sigma_{uv} \chi_f^t(v)|}{\sum_u d_u |\chi_f^t(u)|}$$

$$= \frac{\sum_{\{u,v\}\in E_{\Omega_f(t)}} |\chi_f^t(u) - \sigma_{uv} \chi_f^t(v)| + |\partial \Omega_f(t)|}{\sum_{u\in\Omega_f(t)} d_u}.$$

$$\geq \frac{\min_{\tau:\Omega_f(t)\to\{+1,-1\}} \sum_{\{u,v\}\in E_{\Omega_f(t)}} |1 - \tau(u)\sigma_{uv}\tau(v)| + |\partial\Omega_f(t)|}{\mathrm{vol}(\Omega_f(t))} = \phi^\sigma(\Omega_f(t)).$$

Consider the "integrated" ratio.

$$\frac{\int_0^1 \sum_{\{u,v\}} |\chi_f^t(u) - \sigma_{uv} \chi_f^t(v)|\, dt}{\int_0^1 \sum_u d_u |\chi_f^t(u)|\, dt}$$

For the denomator, we have

$$\int_0^1 |\chi_f^t(u)|\,dt \qquad \left(|\chi_f^t(u)| = \begin{cases} 1, & \text{if } |f(u)|>t \\ 0, & \text{otherwise} \end{cases} \right)$$

$$= \int_0^{|f(u)|} 1\,dt = |f(u)|$$

Hence $\displaystyle\int_0^1 \sum_u d_u |\chi_f^t(u)|\,dt = \sum_u d_u |f(u)|$.

For the numerator, we have

$$\int_0^1 \sum_{\{u,v\}} |\chi_f^t(u) - \sigma_{uv}\chi_f^t(v)|\,dt$$

$$= \sum_{\{u,v\}} \int_0^1 |\chi_f^t(u) - \sigma_{uv}\chi_f^t(v)|\,dt$$

It remains to show:

<u>Claim</u>. $\displaystyle\int_0^1 |\chi_f^t(u) - \sigma_{uv}\chi_f^t(v)|\,dt = |f(u) - \sigma_{uv}f(v)|$.

<u>Proof</u>: It is enough to show

$$\int_0^1 |\chi_f^t(u) - \chi_f^t(v)|\,dt = |f(u) - f(v)|$$

for any $f: V \to [-1, 1]$.

W.l.o.g., we assume $|f(u)| \geq |f(v)|$.

<u>Case 1</u>. $f(u)$, $f(v)$ have different signs, then

$$|\chi_f^t(u) - \chi_f^t(v)| = \begin{cases} 2 & 0 \leq t \leq |f(v)| \\ 1 & |f(v)| < t \leq |f(u)| \\ 0 & \text{if } t \geq |f(u)|. \end{cases}$$

$$\Rightarrow \int_0^1 |\chi_f^t(u) - \chi_f^t(v)|\,dt = 2|f(v)| + |f(u)| - |f(v)|$$

$$= |f(u)| + |f(v)| = |f(u) - f(v)|.$$

<u>Case 2</u>: $f(u)$, $f(v)$ have the same ~~different~~ signs, then

$$|\chi_f^t(u) - \chi_f^t(v)| = \begin{cases} 0, & \text{if } 0 \leq t < |f(v)| \\ 1, & \text{if } |f(v)| \leq t < |f(u)| \\ 0, & \text{if } t > |f(u)| \end{cases}$$

$$\Rightarrow \int_0^1 |\chi_f^t(u) - \chi_f^t(v)|\, dt = |f(u)| - |f(v)| = |f(u) - f(v)| \qquad \square.$$

(13F)

Very roughly to say, our previous proofs uses Lemma II.9.2 applying to f^2. Here we need a more sophisticated version of the square function.

<u>A new "square" function</u> (Kwok, Lau, Lee, Oveis Gharan, Trevisan)

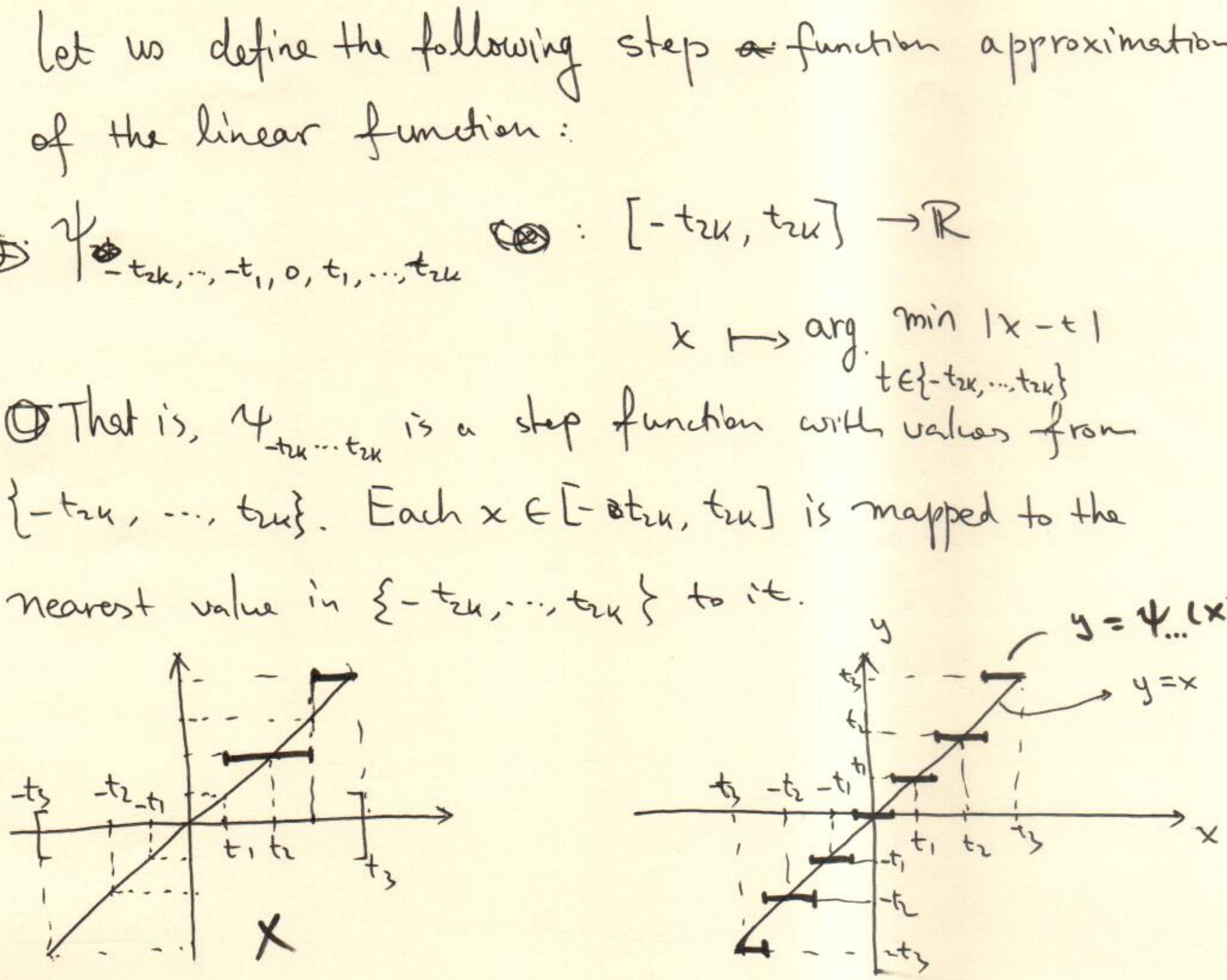
Recall $f(x) = x^2$ can be obtained via $\int_0^x 2t\, dt$, i.e., the integration of a linear function $h(t) = t$.

Let $0 = t_0 \le t_1 \le \cdots \le t_{2k}$ be a sequence of real numbers. Let us define the following step function approximation of the linear function:

$$\psi_{-t_{2k}, \cdots, -t_1, 0, t_1, \cdots, t_{2k}} : [-t_{2k}, t_{2k}] \to \mathbb{R}$$

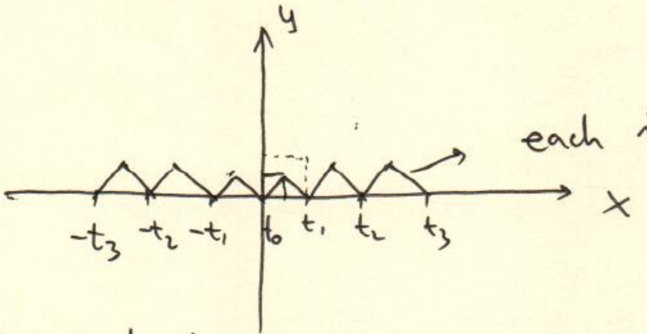
$$x \mapsto \arg \min_{t \in \{-t_{2k}, \cdots, t_{2k}\}} |x - t|$$

That is, $\psi_{-t_{2k} \cdots t_{2k}}$ is a step function with values from $\{-t_{2k}, \cdots, t_{2k}\}$. Each $x \in [-t_{2k}, t_{2k}]$ is mapped to the nearest value in $\{-t_{2k}, \cdots, t_{2k}\}$ to it.



We further define $\eta : [-t_{2k}, t_{2k}] \to \mathbb{R}$

s.t. $\eta(x) := |x - \psi_{\dots}(x)|$



each is a isosceles triangles with hight $\frac{|t_i - t_{i-1}|}{2}$

Particularly, we observe, $\eta(-x) = x$.

Define $F : [-t_{2k}, t_{2k}] \to \mathbb{R}$ to be

$$F(x) = \int_0^x \eta(t)\,dt.$$

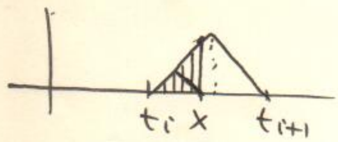
(When $x < 0$, this means $F(x) = -\int_x^0 \eta(t)\,dt$).

This is a monotonic function.

If $x \in [t_i, t_{i+1}]$, for some i, we have

$$F(x) = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \eta(t)\,dt + \int_{t_i}^{x} \eta(t)\,dt$$

$$= \sum_{j=1}^{i-1} \frac{1}{4}(t_{j+1} - t_j)^2 + \int_{t_i}^{x} \eta(t)\,dt$$



$$\geq \sum_{j=1}^{i-1} \frac{1}{4}(t_{j+1} - t_j)^2 + \frac{1}{4}(x - t_i)^2 \qquad (1)$$

That is why we say $F(x)$ is a kind of square function.

In deed.

$$x^2 = \left(\sum_{j=0}^{i-1} (t_{j+1} - t_j) + (x - t_i)\right)^2 \leq i\left(\sum_{j=0}^{i-1} (t_{j+1} - t_j)^2 + (x - t_i)^2\right)$$

$$\leq 2k\left(\sum_{j=0}^{i-1} (t_{j+1} - t_j)^2 + (x - t_i)^2\right) \qquad (2)$$

Combining (1) and (2) leads to

$$\boxed{F(x) \geq \frac{1}{8k} x^2}.$$