Next, we take $t_{2K} = \max_{u \in V} |f(u)|$, for a given $f : V \to \mathbb{R}$.

Lemma II.9.3. For any $\{u, v\} \in E$.

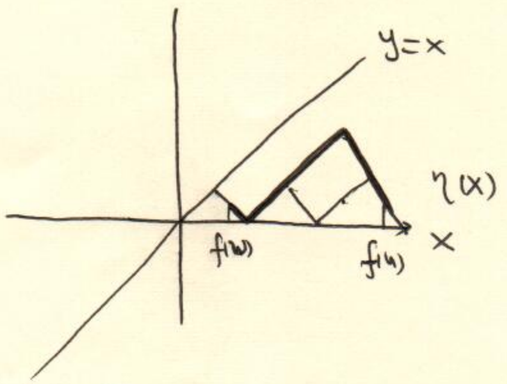
$$|F(f(u)) - \sigma_{uv} F(f(v))|$$
$$\le \frac{1}{2} |f(u) - \sigma_{uv} f(v)| \left(|f(u) - \sigma_{uv} f(v)| + |f(u) - g(u)| + |f(v) - g(v)| \right)$$

where $g(u) := \psi_{-t_{2K}, \dots, t_{2K}}(f(u))$, that is, $|f(u) - g(u)| = \eta(f(u))$

Remark. The rough idea is that $|F(f(u)) - F(f(v))|$

$$= \int_{f(v)}^{f(u)} \eta(x)\,dx \quad , \text{ and } |\eta(x)| \le |f(u) - f(v)| + \varepsilon .$$

(when $f(u), f(v) > 0$)



Proof. Recall $F(f(u)) = \int_0^{f(u)} \eta(x)\,dx$,

$$\sigma_{uv} F(f(v)) = \sigma_{uv} \int_0^{f(v)} \eta(x)\,dx = \int_0^{\sigma_{uv} f(v)} \eta(x)\,dx$$

$\eta(-x) = \eta(x)$.

Moreover,

$$\psi_{-t_{2K}, \dots, t_{2K}}(\sigma_{uv} f(v)) = \sigma_{uv} \psi_{-t_{2K}, \dots, t_{2K}}(f(v)).$$

W.l.o.g., we only need show

$$|F(f(u)) - F(f(v))| \le \frac{1}{2} |f(u) - f(v)| \left(|f(u) - f(v)| + |f(u) - g(u)| + |f(v) - g(v)| \right).$$

For any nontrivial function $f : V \to \mathbb{R}$.

W.l.o.g. we assume $|f(u)| \geq |f(v)|$.

Case(i). $f(u)$, $f(v)$ have different signs, say $f(u) \geqslant 0$, $f(v) \leqslant 0$.

we have

$$|F(u)-F(v)| = \int_0^{f(u)} \eta(x)\,dx - \int_0^{f(v)} \eta(x)\,dx$$

$$\overset{\eta(-x)=\eta(x)}{=} \int_0^{f(u)} \eta(x)\,dx + \int_0^{-f(v)} \eta(x)\,dx$$

$$\leq \int_0^{f(u)} x\,dx + \int_0^{-f(v)} x\,dx$$

$$= \frac{1}{2}\left(f(u)^2 + f(v)^2\right) \underset{f(u)f(v) \leqslant 0.}{\leq} \frac{1}{2}|f(u)-f(v)|^2.$$

Case(ii) $f(u)$, $f(v)$ have the same sign, say $f(u)>0$, $f(v)>0$.

$$|F(u)-F(v)| = \int_{f(v)}^{f(u)} \eta(x)\,dx \leqslant |f(u)-f(v)| \cdot \max_{f(u) \leq x \leq f(v).} \eta(x)$$

Observe that for $f(u) \leq x \leq f(v)$,

$$\eta(x) \leqslant \min\{|x-\psi(f(u))|, |x-\psi(f(v))|\}$$

$$\leq \frac{1}{2}\left(|x-\psi(f(u))| + |x-\psi(f(v))|\right)$$

$$\leq \frac{1}{2}\left(|x-f(u)| + |f(u)-\psi(f(u))| + |x-f(v)| + |f(v)-\psi(f(v))|\right)$$

$$\leq \frac{1}{2}\left(|f(u)-f(v)| + |f(u)-g(u)| + |f(v)-g(v)|\right).$$

Therefore, $|F(u)-F(v)| \leq \frac{1}{2}|f(u)-f(v)|^2 + \frac{1}{2}|f(u)-f(v)| \cdot \begin{pmatrix} |f(u)-g(u)| \\ +|f(v)-g(v)| \end{pmatrix}$

$\square$

<u>Lemma II.9.4</u>. For any nontrivial $f: V \to \mathbb{R}$, and any sequence

$0 = t_0 \leq t_1 \leq \cdots \leq t_{2k} := \max_{u \in V} |f(u)|$, there exists

$\exists\ t_0 \in [0, \max_{u\in V}|f(u)|]$ s.t.

$$\phi^\sigma(\Omega_f(t_0)) \leq 4kR^\sigma(f) + 4\sqrt{2}k\,\frac{\|f-g\|_d}{\|f\|_d}\sqrt{R^\sigma(f)}.$$

Where $g(u) := \psi_{-t_k,\dots,t_k}(f(u))$, $\|f\|_d = \sqrt{\sum_{u\in V} f(u)^2 d_u}$

<u>Proof</u>: For any nontrivial $f: V \to \mathbb{R}$, we consider the function $F\circ f: V \longrightarrow \mathbb{R}$.

By Lemma II.9.2, $\exists\ t \in [0, \max_{u\in V}|F(f(u))|]$ s.t.

$$\phi^\sigma(\Omega_{F\circ f}(t)) \leq \frac{\sum_{\{u,v\}} |F(f(u)) - \sigma_{uv}F(f(v))|}{\sum_{u\in V} d_u |F(f(u))|}$$

where $\Omega_{F\circ f}(t) = \{u\in V : |F(f(u))| > t\}$

Using the fact, $F(x) \geq \frac{1}{8k}x^2$, we derive

$$\sum_{u\in V} d_u|F(f(u))| \geq \frac{1}{8k}\sum_{u\in V} d_u f(u)^2.$$

Therefore,

$$\phi^\sigma(\Omega_{F\circ f}(t)) \leq \frac{\sum_{\{u,v\}} |F(f(u)) - \sigma_{uv}F(f(v))|}{\frac{1}{8k}\sum_{u\in V} f(u)^2 d_u}$$

Applying Lemma II.9.3, we continue to estimate

$$\phi^\sigma(\Omega_{F\circ f}(t)) \leq 8k\,\frac{\sum_{\{u,v\}}\left[\frac{1}{2}|f(u)-\sigma_{uv}f(v)|^2 + \frac{1}{2}|f(u)-\sigma_{uv}f(v)|\left(|f(u)-g(u)| + |f(v)-g(v)|\right)\right]}{\sum_{u\in V} f(u)^2 d_u}$$

$$= 4kR^\sigma(f) + 4k\,\frac{\sum_{\{u,v\}}|f(u)-\sigma_{uv}f(v)|\left(|f(u)-g(u)| + |f(v)-g(v)|\right)}{\sum_{u\in V} f(u)^2 d_u}$$

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$$\leq 4kR^\sigma(f) + 4k\sqrt{R^\sigma(f)}\,\frac{\left(\sum_{\{u,v\}}\left(|f(u)-g(u)| + |f(v)-g(v)|\right)^2\right)^{\frac{1}{2}}}{\|f\|_d}$$

~~Then~~ Notice that,

$$\sum_{\{u,v\}} \left(|f(u) - g(u)| + |f(v) - g(v)| \right)^2$$

$$\leq \sum_{\{u,v\}} \left(2|f(u) - g(u)|^2 + 2|f(v) - g(v)|^2 \right)$$

$$= 2 \sum_{u \in V} \sum_{v : v \sim u} |f(u) - g(u)|^2$$

$$= 2 \sum_{u \in V} d_u |f(u) - g(u)|^2 = \|f - g\|_d^2 .$$

That is, there exists t_0, s.t.

$$\phi^\sigma \left(\Omega_{F \circ f} (t_0) \right) \leq 4k R^\sigma(f) + 4\sqrt{2} k \sqrt{R^\sigma(f)} \frac{\|f - g\|_d}{\|f\|_d} .$$

The Lemma then follows immediately by observing

$$f(u) \geq f(v) \iff F(f(u)) \geq F(f(v)) . \qquad \square$$

~~Let us fix the notation T_0 we found in Lemma II 9.4.~~

Lemma II.9.5. For any nontrivial $f : V \to \mathbb{R}$ and any $1 \leq k \leq |V|$, there exists T_0 as ~~constructed in Lemma II 9.4~~, s.t. at least one of the following estimates holds.

(i) $\phi^\sigma (\Omega_f (T_0)) \leq 8k R^\sigma(f)$.

(ii) there exists k disjointly supported functions $f_1, \ldots, f_k : V \to \mathbb{R}$ s.t. for each i

$$R^\sigma(f_i) < 256 k^2 \frac{R^\sigma(f)^2}{\phi^\sigma (\Omega_f (T_0))^2} .$$

Proof: Denote by $\phi^\sigma(f) := \min_t \phi^\sigma (\Omega_f (t))$.

We do it via the following procedure

Let $M := \max_{u \in V} |f(u)|$. We construct $2k+1$ real numbers

$t_0 \le t_1 \le \cdots \le t_{2k} \le M$ as follows:

Set $t_0 = 0$. Suppose that, we have already fixed $t_0, t_1, \dots, t_{i-1}$. We look for $t_i \in [t_{i-1}, M]$ s.t.

$$\sum_{u: -t_i \le f(u) < -t_{i-1}} d_u \left| f(u) - \Psi_{-t_i, -t_{i-1}}(f(u)) \right|^2$$

$$+ \sum_{u: t_{i-1} < f(u) \le t_i} d_u \left| f(u) - \Psi_{t_{i-1}, t_i}(f(u)) \right|^2 = C \qquad (*)$$

where $C = \dfrac{\phi^\sigma(f)^2 \|f\|_d^2}{256 k^3 R^\sigma(f)}$.

Notice that, LHS is continuous and non-decreasing w.r.t. t_i.

If we can find such constants satisfying $(*)$, we set the smallest of them to be t_i; otherwise, we set $t_i = M$.

This procedure is considered to be successful if $t_{2k} = M$.

If the procedure suceeds, we define $g(u) := \Psi_{-t_{2k}, \dots, t_{2k}}(f(u))$.

Then

$$\|f - g\|_d^2 \le 2kC$$

Applying Lemma II.9.4, we derive

$$\phi^\sigma(f) \le 4kR^\sigma(f) + 4\sqrt{2}k\sqrt{R^\sigma(f)} \, \frac{\sqrt{2k \cdot \dfrac{\phi^\sigma(f)^2 \|f\|_d^2}{256k^3 R^\sigma(f)}}}{\|f\|_d}$$

$$= 4kR^\sigma(f) + \frac{\phi^\sigma(f)}{2}$$

That is, $\phi^\sigma(f) \le 8kR^\sigma(f)$. Hence (i) holds.

If, on the other hand, the procedure fails, that is, we have $t_{2k} < M$, then we define $2k$ disjointly supported functions as

$$f_i(u) = \begin{cases} -|f(u) - \psi_{-t_{i-1}, t_i}(f(u))|, & \text{if } -t_i \le f(u) < -t_{i-1} \\ |f(u) - \psi_{t_{i-1}, t_i}(f(u))|, & \text{if } t_{i-1} < f(u) \le t_i \\ 0, & \text{otherwise.} \end{cases}$$

for $i = 1, 2, \cdots, k$. By construction, we have

$$\|f_i\|_d^2 = C.$$

<u>Claim</u>: For any $\{u,v\} \in E$,

$$\sum_{i=1}^{2k} |f_i(u) - \sigma_{uv} f_i(v)|^2 \le |f(u) - \sigma_{uv} f(v)|^2.$$

<u>Proof</u>: Noticing that if we define $h(u) = f(u)$, $h(v) = \sigma_{uv} f(v)$.

Then $h_i(u) = f_i(u)$, $h_i(v) = \sigma_{uv} f_i(v)$.

Therefore, it is enough to consider the case $\sigma_{uv} = +1$.

<u>Case 1</u>. u, v lie in the support of the same fct f_i.

$$\sum_{j=1}^{2k} |f_j(u) - f_j(v)|^2 = |f_i(u) - f_i(v)|^2$$

If $f(u)f(v) \ge 0$, say $f(u) \ge 0$, $f(v) \ge 0$, then

$$|f_i(u) - f_i(v)|^2 = \Big|\,|f(u) - \psi_{t_{i-1},t_i}(f(u))| - |f(v) - \psi_{t_{i-1},t_i}(f(v))|\,\Big|^2$$

W.l.o.g., suppose $|f(u) - \psi_{t_{i-1},t_i}(f(u))| \ge |f(v) - \psi_{t_{i-1},t_i}(f(v))|$

we have $|f(u) - \psi_{t_{i-1},t_i}(f(u))| - |f(v) - \psi_{t_{i-1},t_i}(f(v))|$

$$\le |f(u) - \psi_{t_{i-1},t_i}(f(v))| - |f(v) - \psi_{t_{i-1},t_i}(f(v))|$$

$$\le |f(u) - f(v)|.$$

Hence, $|f_i(u) - f_i(v)|^2 \le |f(u) - f(v)|^2$.

→ If $f(u)f(v)<0$, say $f(u)>0$, $f(v)<0$.

$$|f_i(u)-f_i(v)|^2 = \Big|\,|f(u)-\psi_{t_{i-1},t_i}(f(u))| + |f(v)-\psi_{-t_{i0},-t_{i1}}(f(v))|\,\Big|^2$$

$$= \Big|\,|f(u)-\psi_{t_{i-1},t_i}(f(u))| + |-f(v)-\psi_{t_{i-1},t_i}(-f(v))|\,\Big|^2$$

$$\leq \Big|\,|f(u)-t_{i-1}| + |-f(v)-t_{i-1}|\,\Big|^2 \leq |f(u)-f(v)|^2.$$

<u>Case 2</u>. $u\in supp(f_i)$, $v\in supp(f_j)$, $i\neq j$. We can assume $j>i$.

Then $\sum_{i=1}^{2k} |f_i(u)-f_i(v)|^2 = |f_i(u)|^2+|f_j(v)|^2$.

→ If $f(u)f(v)\geq 0$, say $f(u)\geq 0$, $f(v)\geq 0$, then

$$|f_i(u)|^2+|f_j(v)|^2 = |f(u)-\psi_{t_{i-1},t_i}(f(u))|^2 + |f(v)-\psi_{t_{j-1},t_j}(f(v))|^2$$

$$\leq |f(u)-t_i|^2 + |f(v)-t_i|^2$$

$$\leq |f(u)-f(v)|^2.$$

→ If $f(u)f(v)<0$, say, $f(u)>0$, $f(v)<0$, then

$$|f_i(u)|^2+|f_j(v)|^2 = |f(u)-\psi_{t_{i-1},t_i}(f(u))|^2 + |f(v)-\psi_{-t_j,-t_{j-1}}(f(v))|^2$$

$$= |f(u)-\psi_{t_{i-1},t_i}(f(u))|^2 + |-f(v)-\psi_{t_{j-1},t_j}(-f(v))|^2$$

$$\leq |f(u)-t_{i-1}|^2 + |-f(v)-t_{j-1}|^2 \leq |f(u)-f(v)|^2.$$

This completes the proof of the Claim.

With the claim in hand, we calculate

$$\sum_{i=1}^{2k} R^\sigma(f_i) = \frac{1}{C}\sum_{i=1}^{2k}\sum_{\{u,v\}} (f_i(u)-\sigma_{uv}f_i(v))^2$$

$$\leq \frac{1}{C}\sum_{\{u,v\}} |f(u)-\sigma_{uv}f(v))^2$$

$$= 256k^3\frac{R^\sigma(f)^2}{\phi^\sigma(f)^2} := \mathcal{C}k^3, \text{ where } \mathcal{C} = \frac{256R^\sigma(f)^2}{\phi^\sigma(f)^2}$$

Then we can find k fcts from the set $\{f_1, f_2, \cdots, f_{2k}\}$, relabeling them as $f_1, f_2, \cdots, f_k$, s.t. $R^\sigma(f_i) < ek^2$, $i=1,\cdots,k$.

Since otherwise, there exists $k+1$ fcts with $R^\sigma(f) \geq ek^2$. This tells $\sum_{i=1}^{2k} R^\sigma(f_i) \geq ek^2(k+1) > ek^3$, a contradiction. □

Theorem Given a signed graph $\Gamma = (G, \sigma)$ and any $k \in \{1, 2, \cdots, N\}$, at least one of the following holds.

(i) $h_1^\sigma \leq 8k \lambda_1^\sigma$

(ii) $h_1^\sigma < 16\sqrt{2}\, k \dfrac{\lambda_1^\sigma}{\sqrt{\lambda_k^\sigma}}$.

Proof. This is a direct consequence of Lemma II.9.5. and Lemma II.9.6 below. □

Lemma II.9.6. For any k disjointly supported fcts $f_1, \cdots, f_k : V \to \mathbb{R}$,

$$\lambda_k \leq 2 \max_{1\leq i\leq k} R^\sigma(f_i).$$

Proof By min-max principle, it sufficient to show for any $h \in \text{span}\{f_1, \cdots, f_k\}$, $R(h) \leq 2\max_{1\leq i\leq k} R(f_i)$

Let $h = \sum_{i=1}^k c_i f_i$, we have

$$|h(u) - \sigma_{uv} h(v)|^2 = \Big|\sum_{i=1}^k c_i f_i(u) - \sigma_{uv} \sum_{i=1}^k c_i f_i(v)\Big|^2$$
$$= \Big|\sum_{i=1}^k \left(c_i f_i(u) - \sigma_{uv} c_i f_i(v)\right)\Big|^2$$
$$\leq \sum_{i=1}^k 2\, |c_i f_i(u) - \sigma_{uv} c_i f_i(v)|^2$$

Therefore,

$$R^\sigma(h) = \frac{\sum_{\{u,v\}} (h(u) - \sigma_{uv} h(v))^2}{\sum_u h(u)^2 d_u} \leq \frac{2\sum_{i=1}^{k} \sum_{\{u,v\}} (c_i f_i(u) - \sigma_{uv} c_i f_i(v))^2}{\sum_u \sum_{i=1}^{k} c_i^2 f_i^2(u) d_u}$$

$$\leq 2 \max_{1 \leq i \leq k} R^\sigma(f_i). \qquad \square.$$

<u>Remark</u>. Observing $\sqrt{\frac{2}{\lambda_k^\sigma}} \geq 1$, Theorem II.9.1. follows directly from Theorem ✱.