Therefore,

$$R^\sigma(h) = \frac{\sum_{\{u,v\}} (h(u) - \sigma_{uv} h(v))^2}{\sum_u h(u)^2 d_u} \leq \frac{2\sum_{i=1}^{k} \sum_{\{u,v\}} (c_i f_i(u) - \sigma_{uv} c_i f_i(v))^2}{\sum_u \sum_{i=1}^{k} c_i^2 f_i^2(u) d_u}$$

$$\leq 2 \max_{1 \leq i \leq k} R^\sigma(f_i). \qquad \square.$$

<u>Remark</u>: Observing $\sqrt{\frac{2}{\lambda_k^\sigma}} \geq 1$, Theorem II.9.1. follows directly from Theorem $(\ast)$.

We also have a higher order version of Thm II.9.1 and Thm II.9.6 $(\ast)$.

<u>Theorem II.9.8</u> There exists an absolute constant C such that for any signed graph $\Gamma = (G, \sigma)$ and any $1 \leq k \leq l \leq |V|$, at least one of the two estimates holds:

(i) $h_k^\sigma \leq 8 C l k^6 \lambda_k^\sigma$.

(ii) $h_k^\sigma < 16\sqrt{2} C l k^6 \dfrac{\lambda_k^\sigma}{\sqrt{\lambda_l^\sigma}}$.

As a consequence, for any $\Gamma = (G, \sigma)$ and any $1 \leq k \leq l \leq |V|$ we have

$$h_k^\sigma < 16\sqrt{2} C l k^6 \frac{\lambda_k^\sigma}{\sqrt{\lambda_l^\sigma}}.$$

<u>Proof</u>: We need the following fact from random partition theory: For any $k \in \{1, 2, \cdots, |V|\}$, there exists k disjointly supported functions $\psi_1, \psi_2, \cdots, \psi_k : V \to \mathbb{R}$ s.t.

$$R^\sigma(\psi_i) \leq C k^6 R^\sigma(\Phi) \leq C k^6 \lambda_k^\sigma, \quad i = 1, 2, \cdots, k, \quad (\ast\ast)$$

where C is an absolute constant. Recall Φ is the map from V to $\mathbb{R}^k$ given by the first k eigenfunctions.

Next, we apply Lemma II.9.5 to each ψ_i: For each ψ_i, for any $k \leq \ell \leq |V|$, at least one of the following 2 holds:

(i) $\phi^\sigma(\psi_i) \leq 8\ell R^\sigma(\psi_i)$

(ii) $\dfrac{\lambda_\ell^\sigma}{2} < 256\ell^2 \dfrac{R^\sigma(\psi_i)^2}{\phi^\sigma(\psi_i)^2}$.

Note that, we already apply Lemma II.9.7 in (ii) for the ease of notation. Since ψ_i's are disjointly supported, we observe that

$$h_k^\sigma \leq \phi^\sigma(\psi_i), \quad \forall i=1,2,\cdots,k.$$

We then further apply the fact (**). and derive. at least one of the two holds:

(i) $h_k^\sigma \leq 8\ell \cdot Ck^6 \lambda_k^\sigma$ (ii) $\dfrac{\lambda_\ell^\sigma}{2} < 256\ell^2 \dfrac{(Ck^6\lambda_k^\sigma)^2}{(h_k^\sigma)^2}$.

That is, at least one of the following two holds:

(i) $h_k^\sigma \leq 8C\ell k^6 \lambda_k^\sigma$, (ii) $h_k^\sigma < 16\sqrt{2}\, C\ell k^6 \dfrac{\lambda_k^\sigma}{\sqrt{\lambda_\ell^\sigma}}$.

Using the fact $\lambda_\ell^\sigma \leq 2$, (i) implies

$$h_k^\sigma \leq 8\sqrt{2}\, C\ell k^6 \frac{\lambda_k^\sigma}{\sqrt{\lambda_\ell^\sigma}}.$$

□.

(II.10) Buser inequality

In last section, we discuss when the lower bound esti. of Cheeger inequalities have the same order as its upper bound estimates. In this section, we consider a "dual" question: When does the upper bound estimates have the same order as its lower bounds? That is, under which conditions can it holds that

$$\lambda_k^\sigma \lesssim (h_k^\sigma)^2 ?$$

One ~~such~~ answers in this direction is Buser type inequalities, imposing certain curvature ~~conditions~~ restrictions of the underlying graph. Our aim is to show the following:

<u>Theorem II 10.1</u>: For any signed graph $\Gamma = (G, \sigma)$ satisfying $CD^\sigma(0,\infty)$ condition, and any $k \in \{1, 2, \ldots, |V|\}$, we have

$$\lambda_k^\sigma \leq 16 k^2 \log(2k) \, d_{max} (h_k^\sigma)^2 .$$

(L., Münch, Peyerimhoff. SIAM J. Discrete Math. 33 (2019), no. 1, 257-305)

<u>Curvature of a signed graph</u>.

The <u>motivation</u> for the curvature-type condition $CD^\sigma(0,\infty)$ ~~is~~ lies in the Bochner identity in Riemannian geometry. Recall on a Riemannian manifold (M^n, g), for any $f \in C^\infty(M)$, we have

$$\frac{1}{2}\Delta |\nabla f|^2 = |\mathrm{Hess} f|^2 + \langle \nabla \Delta f, \nabla f \rangle + Ric(\nabla f, \nabla f).$$

The Ricci curvature term $Ric(\nabla f, \nabla f)$ appears when one (148)
try to commute the order of taking covariant derivatives using
Ricci identity. By Cauchy-Schwarz inequality, we have

$$|\mathrm{Hess} f|^2 \geq \frac{1}{n}(\Delta f)^2.$$

Indeed, we have the following characteristic of Ric curvature lower bound:

On (M^n, g), let $x \in M$, then $Ric \geq K$ at x iff

$$(*) \quad \frac{1}{2}\Delta|\nabla f|^2_{(x)} \geq \frac{1}{n}(\Delta f)^2_{(x)} + \langle \nabla \Delta f, \nabla f\rangle_{(x)} + K|\nabla f|^2_{(x)} \text{ holds at } x$$

for all $f \in C^\infty(M)$.

Then a key observation due to Bakry and Émery in 1980s is as follows: One can rewrite $(*)$ as

$$\frac{1}{2}\Delta|\nabla f|^2(x) - \langle \nabla\Delta f, \nabla f\rangle_{(x)} \geq \frac{1}{n}(\Delta f)^2_{(x)} + K|\nabla f|^2(x). \quad (1)$$

Compare it with the classical chain rule formula:

$$\frac{1}{2}\Delta(f^2)_{(x)} - f\Delta f(x) = |\nabla f|^2(x). \quad (2)$$

The LHS$_s$ of (1) and (2) share the same pattern.

<u>Definition</u> II.10.2: (Bakry - Émery) For any $f, g \in C^\infty(M)$, we define

$$\Gamma(f,g)(x) := \frac{1}{2}\left(\Delta(fg)(x) - f(x)\Delta g(x) - \Delta f(x)\cdot g(x)\right)$$

and $\Gamma_2(f,g)(x) := \frac{1}{2}\left(\Delta\Gamma(f,g)(x) - \Gamma(f,\Delta g)(x) - \Gamma(\Delta f, g)(x)\right)$.

<u>Remark</u>: One may denote $\Gamma_0(f,g)(x) := f(x)g(x)$. Then $\Gamma_2(f,g)(x)$ is obtained by replacing Γ_0 by Γ in the definition of $\Gamma(f,g)(x)$.

In principle, one can go on ~~&~~ definining $\Gamma_3, \Gamma_4, \cdots$. $\square$

Observe that, LHS of (2) is $\Gamma(f,f)(x) := \Gamma(f)(x)$, while the LHS of (1) is $\Gamma_2(f,f)(x) := \Gamma_2(f)(x)$.

The characterization of $Ric \geq K$ at x in (M^n, g) can be reformulated as below:

$(*)'$ $\Gamma_2(f)(x) \geq \frac{1}{n}(\Delta f)^2 + K\,\Gamma(f)(x)$, holds at x for any $f \in C^\infty(M)$.

In this way, one can talk about "curvature" of an operator.

$(*)'$ is also called the curvature-dimension inequality $CD(K,n)$.

Particularly, we can define $CD(K,n)$ for Laplacian on a graph.

<u>Definition</u> II.10.3. Let $G = (V,E)$ be a graph. Γ, Γ_2 are defined as in Definition II.10.2 via the graph Laplacian. Then ~~G is said~~

We say G satisfies $CD(K,n)$ at $x \in V$ if

$$\Gamma_2(f)(x) \geq \frac{1}{n}(\Delta f(x))^2 + K\,\Gamma(f)(x), \quad \text{for any } f: V \to \mathbb{R}.$$

<u>Prop II.10.4.</u> For any $f: V \to \mathbb{R}$, we have at $x \in V$

$$\Gamma(f)(x) = \frac{1}{2}\frac{1}{d_x}\sum_{y: y\sim x}(f(y) - f(x))^2$$

<u>Proof</u>. This follows directly by definition:

$$\begin{aligned}
\Gamma(f)(x) &= \frac{1}{2}\Delta(f^2)(x) - f(x)\Delta f(x) \\
&= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\left(f^2(y) - f^2(x)\right) - f(x)\cdot\frac{1}{d_x}\sum_{y:y\sim x}(f(y) - f(x)) \\
&= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\left(f^2(y) - f^2(x) - 2f(x)f(y) + 2f^2(x)\right) \\
&= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}(f(y) - f(x))^2 .
\end{aligned}$$

$\square$

Rmk. This fits with our previous discussions very well. For example,

$$\sum_{\{x,y\}} (f(y) - f(x))^2 = \frac{1}{2} \sum_{x\in V} \sum_{y: y\sim x} (f(y) - f(x))^2$$

$$= \sum_{x\in V} \frac{1}{2d_x} \sum_{y:y\sim x} (f(y) - f(x))^2 d_x$$

$$= \sum_{x\in V} \Gamma(f)(x) d_x .$$

The Rayleigh quotient

$$R(f) = \frac{\sum_{x\in V} \Gamma(f)(x) d_x}{\sum_{x\in V} f(x)^2 d_x} . \qquad \square$$

Next, we ~~look~~ for ~~something~~ formulation of Γ, Γ_2 which fits neatly with the signed Laplacian! The Most natural idea is to use Δ^σ instead of Δ in the definitions of Γ, Γ_2, and $CD(K,n)$. However, this does not work. The proper way to do it is as follows.

Definition II.10.5. (Curvature-dimension ineq. for signed Laplacian) For any two functions $f, g: V \longrightarrow \mathbb{R}$, we define at $x\in V$

$$\Gamma^\sigma(f,g)(x) := \frac{1}{2}\left(\Delta(fg)(x) - f(x)\Delta^\sigma g(x) - \Delta^\sigma f(x)\cdot g(x)\right),$$

and $\Gamma_2^\sigma(f,g)(x) := \frac{1}{2}\left(\Delta\Gamma^\sigma(f,g)(x) - \Gamma^\sigma(f,\Delta^\sigma g)(x) - \Gamma^\sigma(\Delta^\sigma f, g)(x)\right)$.

We say a signed graph $\Gamma = (G,\sigma)$ satisfies ~~the~~ $CD^\sigma(K,n)$ at $x\in V$ if

$$\Gamma_2^\sigma(f)(x) \geq \frac{1}{n}\left(\Delta^\sigma f(x)\right)^2 + K\Gamma^\sigma(f)(x), \text{ holds for any } f: V\to\mathbb{R}.$$

Prop II.10.6. For any $f: V\to\mathbb{R}$, we have at $x\in V$

$$\Gamma^\sigma(f)(x) = \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}(\sigma_{xy}f(y) - f(x))^2 .$$

Proof: This again follows directly from definition:

$$\Gamma^\sigma(f)(x) = \frac{1}{2}\Delta(f^2)(x) - f(x)\Delta^\sigma f(x)$$

$$= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\left(f^2(y)-f^2(x)\right) - f(x)\cdot\frac{1}{d_x}\sum_{y:y\sim x}\left(\sigma_{xy}f(y)-f(x)\right)$$

$$= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\left(f^2(y)-f^2(x)-2\sigma_{xy}f(x)f(y)+2f^2(x)\right)$$

$$= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\left(\sigma_{xy}f(y)-f(x)\right)^2$$

□

Rmk: The corresponding Rayleigh quotient can be reformulated as:

$$R^\sigma(f) = \frac{\sum_{x\in V}\Gamma^\sigma(f)(x)\,d_x}{\sum_{x\in V} f(x)^2\,d_x}.$$

Prop II.10.7: The CD^σ inequality is switching invariant. Let $\Gamma=(G,\sigma)$ be a signed graph. Let $\tau: V\to\{+1,-1\}$ be a switching function. Then (G,σ^τ) satisfies $CD^{\sigma^\tau}(K,n)$ iff (G,σ) satisfies $CD^\sigma(K,n)$.

Proof: Observe that we have as matrices (Recall $\sigma^\tau_{xy}=\tau(x)\sigma_{xy}\tau(y)$).

$$\Delta^{\sigma^\tau} = \begin{pmatrix}\tau(x_1) & & \\ & \ddots & \\ & & \tau(x_n)\end{pmatrix}\Delta^\sigma\begin{pmatrix}\tau(x_1) & & \\ & \ddots & \\ & & \tau(x_n)\end{pmatrix}$$

$$=: M_\tau^{-1}\Delta^\sigma M_\tau.$$

Furthermore, for any $f, g: V\to\mathbb{R}$, we have

$$\Gamma^{\sigma^\tau}(f,g)=\Gamma^\sigma(\tau f,\tau g),\quad \Gamma_2^{\sigma^\tau}(f,g)=\Gamma_2^\sigma(\tau f,\tau g).$$

Then the proposition follows immediately. □

Rmk: $CD^\sigma(K,n)$ at x is a local property: Only the structure of $B_2(x)$ is related to ~~the~~ $CD^\sigma(K,n)$ property at x.

Heat semigroup characterizations

We discuss a very useful characterization of $CD^\sigma(K,\infty)$ for our later purpose.

Consider the following continuous-time heat equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta^\sigma u(x,t) \\ u(x,0) = f(x) \end{cases} \tag{3}$$

where $f: V \to \mathbb{R}$ is an initial function.

The equation (3) has a unique solution according to the Picard-Lindelöf theorem. Applying Picard iteration, we know this unique solution ~~can be~~ is given by the matrix exponential.

$$u(x,t) = \big(e^{t\Delta^\sigma} \underbrace{f}_{\text{considered as a } |V|\text{-vector}}\big)(x).$$

- A brief recall of matrix exponential:

(a) For any square matrix A,

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}, \quad \text{where } A^0 = I.$$

The série is always converge. (entry-wise).

(b) $\frac{d}{dt} e^{At} = A \cdot e^{At}$.

(c) $e^{0} = I$. (0: zero matrix.)

(d) $e^A \cdot e^B = e^{A+B}$ if $AB = BA$.

(e) $Be^{A} = e^A \cdot B$ if $AB = BA$.

(f) $(e^A)^T = e^{A^T}$

(g) $e^A = P^{-1} e^{PAP^{-1}} P$, for P invertible.

For diagonal matrices, the matrix exponential has explicit expression:

$$e^{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}. \quad \square$$

<u>Prop II.10.8</u>: The operator $P_t^\sigma := e^{t\Delta^\sigma}$, $t \geq 0 : \ell^2(V) \to \ell^2(V)$ satisfies the following properties:

(i) P_t^σ is self-adjoint.

(ii) $P_t^\sigma \Delta^\sigma = \Delta^\sigma P_t^\sigma$

(iii) $P_t^\sigma P_s^\sigma = P_{t+s}^\sigma$ for any $t, s \geq 0$.

<u>Proof</u>. (e) $\Rightarrow$ (ii), (d) $\Rightarrow$ (iii), (f) with a modification $\Rightarrow$ (i). $\square$

We also collect ~~s~~ special basic properties of $P_t := e^{t\Delta}$:

<u>Prop II.10.9</u>. (i) All matrix entries of P_t are ~~real and~~ nonnegative.

(ii) $P_t c = c$ for any constant function c.

<u>Proof</u>. Recall all off-digonal entries of Δ are ≥ 0. ($-\Delta$ is a generalized Laplacian).

Hence, $\exists\, C > 0$ s.t. $\Delta + C \cdot I$ is entry-wise ≥ 0.

Then $e^{\Delta + C\cdot I}$ is also entry-wise ≥ 0.

This tells $P_t = e^{\Delta + C\cdot I} \cdot e^{-CI}$ is entry-wise ≥ 0.

For a constant function c, we have $\Delta c = 0$.

Hence $\frac{\partial}{\partial t} P_t c = \Delta c = 0$. This means $P_t c = P_0 c = c$. $\square$

<u>Rmk</u>. Prop. II.10.9 tells that for any function $f: V \to \mathbb{R}$ satisfying $0 \leq f(x) \leq c$ for all $x \in V$, we have $0 \leq P_t f(x) \leq c$, $\forall x \in V$.