<u>Prop II.10.10</u> (Spectral decomposition) Let $\phi_1, \phi_2, \dots, \phi_N$ be an ~~orthogonal~~ orthonormal basis of $\ell^2(V)$ of eigenfunctions of $-\Delta^\sigma$.

Then
$$P_t^\sigma = e^{t\Delta^\sigma} = \sum_{i=1}^{N} e^{-\lambda_i t} I_i$$
where $I_i f = \cancel{D}\langle f, \phi_i\rangle \phi_i$ is the projection operator.

In particular, for any $x, y \in V$,
$$\underset{\substack{\uparrow \\ (x,y)\text{-entry of the matrix } P_t^\sigma}}{P_t^\sigma(x,y)} = \sum_{i=1}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y).$$

<u>Proof</u>: First observe that
$$D^{\frac{1}{2}}\cdot\phi_i = \begin{pmatrix} \sqrt{d_{x_1}} & & \\ & \ddots & \\ & & \sqrt{d_{x_N}} \end{pmatrix}\begin{pmatrix} \phi_i(x_1) \\ \vdots \\ \phi_i(x_N)\end{pmatrix} = \begin{pmatrix} \sqrt{d_{x_1}}\,\phi_i(x_1) \\ \vdots \\ \sqrt{d_{x_N}}\,\phi_i(x_N)\end{pmatrix}, \quad i = 1, \dots, N$$

forms a system of <u>Euclidean</u>-orthonormal basis of $\mathbb{R}^N$.

Let us denote
$$P := D^{\frac{1}{2}}\begin{pmatrix} | & & | \\ \phi_1 & \cdots & \phi_N \\ | & & | \end{pmatrix}.$$

Then P is an orthogonal matrix.

Moreover, $D^{\frac{1}{2}}\phi_i$, $i=1,\dots,N$ ~~is~~ are eigenfunctions of the symmetric matrix $D^{\frac{1}{2}}\Delta^\sigma D^{-\frac{1}{2}}$.

That is,
$$P^T D^{\frac{1}{2}}\Delta^\sigma D^{-\frac{1}{2}} P = \begin{pmatrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_N \end{pmatrix}$$

Hence by property (g) of the matrix exponential, we have

$$P_t^\sigma = e^{t\Delta^\sigma} = D^{-\frac{1}{2}} P \; e^{P^T D^{\frac{1}{2}} t\Delta^\sigma D^{-\frac{1}{2}} P} \cdot P^T D^{\frac{1}{2}}$$

$$= \begin{pmatrix} | & & | \\ \phi_1 & \cdots & \phi_N \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} e^{-\lambda_1 t} & & \\ & \ddots & \\ & & e^{-\lambda_N t} \end{pmatrix} \cdot \begin{pmatrix} - \phi_1 - \\ \vdots \\ - \phi_N - \end{pmatrix} \begin{pmatrix} \sqrt{d_{x_1}} & & \\ & \ddots & \\ & & \sqrt{d_{x_N}} \end{pmatrix}^2$$

$$= \begin{pmatrix} & & \\ & \sum_{i=1}^N \cancel{\phi} e^{-\lambda_i t} \phi_i(x_k) \phi_i(x_l) & \\ & & \end{pmatrix} \begin{pmatrix} d_{x_1} & & \\ & \ddots & \\ & & d_{x_N} \end{pmatrix}$$

$$= \sum_{i=1}^N e^{-\lambda_i t} \begin{pmatrix} \phi_i(x_1)\phi_i(x_1) & \cdots & \phi_i(x_1)\phi_i(x_N) \\ \vdots & \ddots & \vdots \\ \phi_i(x_N)\phi_i(x_1) & \cdots & \phi_i(x_N)\phi_i(x_N) \end{pmatrix} \begin{pmatrix} d_{x_1} & & \\ & \ddots & \\ & & d_{x_N} \end{pmatrix}$$

$$= \sum_{i=1}^N e^{-\lambda_i t} \begin{pmatrix} \phi_i(x_1)\phi_i(x_1) d_{x_1} & \cdots & \phi_i(x_1)\phi_i(x_N) d_{x_N} \\ \vdots & \ddots & \vdots \\ \phi_i(x_N)\phi_i(x_1) d_{x_1} & \cdots & \phi_i(x_N)\phi_i(x_N) d_{x_N} \end{pmatrix}$$

$$\triangleq: \sum_{i=1}^N e^{-\lambda_i t} I_i$$

where for any $f: V \to \mathbb{R}$,

$$I_i f = I_i \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^N \phi_i(x_1)\phi_i(x_k) f(x_k) d_{x_k} \\ \vdots \\ \sum_{k=1}^N \phi_i(x_N)\phi_i(x_k) f(x_k) d_{x_k} \end{pmatrix}$$

$$= \begin{pmatrix} \phi_i(x_1) \\ \vdots \\ \phi_i(x_N) \end{pmatrix} \cdot \sum_{k=1}^N \phi_i(x_k) f(x_k) d_{x_k} = \phi_i \cdot \langle \phi_i, f \rangle .$$

We have the following characterization:

Theorem II.10.1: Let $\Gamma = (G, \sigma)$ be given. TFAE.

(i) $CD^\sigma(K,\infty)$ holds, i.e.

$$\Gamma_2^\sigma(f) \geq K\,\Gamma^\sigma(f), \quad \forall\, f: V \to \mathbb{R}.$$

(ii) $$\Gamma^\sigma(P_t^\sigma f) \leq e^{-2Kt} P_t(\Gamma^\sigma(f)), \quad \forall\, f: V \to \mathbb{R},\ \forall\, t \geq 0$$

(iii) $$P_t(f^2) - (P_t^\sigma f)^2 \geq \frac{1}{K}(e^{2Kt}-1)\,\Gamma^\sigma(P_t^\sigma f) \quad \forall\, f: V \to \mathbb{R},\ \forall\, t \geq 0$$

where $\frac{e^{2Kt}-1}{K}$ reads as $2t$ in the case $K = 0$.

Proof: We will show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). For any $0 \leq s \leq t$, we consider

$$F(s) := e^{-2Ks} P_s\left(\Gamma^\sigma(P_{t-s}^\sigma f)\right)$$

Observe that
$$\begin{cases} F(0) = \Gamma^\sigma(P_t^\sigma f) \\ F(t) = e^{-2Kt} P_t(\Gamma^\sigma(f)) \end{cases}$$

It is sufficient to show $\frac{d}{ds} F(s) \geq 0$.

We calculate

$$\frac{d}{ds} F(s) = -2K e^{-2Ks} P_s(\Gamma^\sigma(P_{t-s}^\sigma f)) + e^{-2Ks} \overset{P_s \Delta}{\overset{\parallel}{\widetilde{\Delta P_s}}}(\Gamma^\sigma(P_{t-s}^\sigma f))$$

$$+ e^{-2Ks} P_s\left(\frac{d}{ds}\Gamma^\sigma(P_{t-s}^\sigma f)\right)$$

where $$\frac{d}{ds}\Gamma^\sigma(P_{t-s}^\sigma f) = \frac{d}{ds}\left(\Gamma^\sigma(P_{t-s}^\sigma f, P_{t-s}^\sigma f)\right)$$

$$= \Gamma^\sigma(-\Delta^\sigma P_{t-s}^\sigma f, P_{t-s}^\sigma f) + \Gamma^\sigma(P_{t-s}^\sigma f, -\Delta^\sigma P_{t-s}^\sigma f)$$

Therefore, we derive

$$\frac{d}{ds}F(s) = 2e^{-2ks} P_s \left(\frac{1}{2}\left[\Delta \Gamma^\sigma (P^\sigma_{t-s} f) - \Gamma^\sigma (\Delta^\sigma P^\sigma_{t-s} f, P^\sigma_{t-s} f) - \Gamma^\sigma (P^\sigma_{t-s} f, \Delta^\sigma P^\sigma_{t-s} f) \right] - k\Gamma^\sigma (P^\sigma_{t-s} f) \right)$$

$$= 2e^{-2ks} P_s \left(\Gamma_2^\sigma (P^\sigma_{t-s} f) - k\Gamma^\sigma (P^\sigma_{t-s} f) \right) \geq 0.$$

where we use in the last step the assumption (i) and the nonnegativity of P_s.

<u>(ii)⇒(iii)</u> For $0 \leq s \leq t$, we consider

$$G(s) := P_s\left((P^\sigma_{t-s} f)^2 \right).$$

<u>Observe that</u>: $\begin{cases} G(0) = (P^\sigma_t f)^2 \\ G(t) = P_t(f^2) \end{cases}$

We calculate

$$\frac{d}{ds}G(s) = \overbrace{\Delta P_s}^{P_s\Delta} ((P^\sigma_{t-s} f)^2) + P_s\left(-2P^\sigma_{t-s} f \cdot \Delta^\sigma P^\sigma_{t-s} f\right)$$

$$= 2P_s \left(\frac{1}{2}\left[\Delta (P^\sigma_{t-s} f)^2 - 2P^\sigma_{t-s} f \Delta^\sigma P^\sigma_{t-s} f \right] \right)$$

$$= 2P_s\left(\Gamma^\sigma (P^\sigma_{t-s} f) \right)$$

$$\overset{(ii)}{\geq} 2e^{2ks} \Gamma^\sigma (P^\sigma_s P^\sigma_{t-s} f)$$

$$= 2e^{2ks} \Gamma^\sigma (P^\sigma_t f)$$

Therefore, we derive

$$G(t) - G(0) = \int_0^t \frac{d}{ds}G(s)\, ds \geq \int_0^t 2e^{2ks} \Gamma^\sigma (P^\sigma_t f)\, ds$$

$$= \Gamma^\sigma (P^\sigma_t f) \cdot \int_0^t 2e^{2ks}\, ds.$$

~~(iii) following by~~ Combining with the following simple fact yields (iii):

$$\int_0^t 2e^{2Ks}\,ds = \begin{cases} 2t, & K=0. \\ \dfrac{e^{2Kt}-1}{K}, & K\neq 0. \end{cases}$$

(iii) $\Rightarrow$ (i). Consider the ineq. (iii) at $t=0$:

$$P_t(f^2) - (P_t^\sigma f)^2 \geq \frac{e^{2Kt}-1}{K}\,\Gamma^\sigma(P_t^\sigma f).$$

Inserting the expansion

$$P_t^\sigma = \sum_{n=0}^\infty \frac{(t\Delta^\sigma)^n}{n!} = I + t\Delta^\sigma + \frac{t^2}{2}(\Delta^\sigma)^2 + \cdots$$

and

$$e^{2Kt} = \sum_{n=0}^\infty \frac{(2Kt)^n}{n!} = 1 + 2Kt + \frac{1}{2}(2Kt)^2 + \cdots$$

~~We obtain~~ yields that

$$\begin{aligned} \text{LHS} &= f^2 + t\Delta(f^2) + \frac{t^2}{2}\Delta^2(f^2) + \cdots \\ &\quad - \left(f + t\Delta^\sigma f + \frac{t^2}{2}(\Delta^\sigma)^2 f + \cdots\right)^2 \\ &= f^2 + t\Delta(f^2) + \frac{t^2}{2}\Delta^2(f^2) + o(t^2) \\ &\quad - f^2 - 2tf\Delta^\sigma f - t^2(\Delta^\sigma f)^2 - t^2 f\cdot(\Delta^\sigma)^2 f + o(t^2) \\ &= 2t\underbrace{\left(\frac{1}{2}\Delta(f^2) - f\Delta^\sigma f\right)}_{\Gamma^\sigma(f)} + \frac{t^2}{2}\Delta^2(f^2) - t^2(\Delta^\sigma f)^2 - t^2\cdot f(\Delta^\sigma)^2 f + o(t^2) \end{aligned}$$

and

$$\text{RHS} = \frac{2Kt + \frac{(2Kt)^2}{2} + o(t^2)}{K}\cdot \Gamma^\sigma\Big(f + t\Delta^\sigma f + \frac{t^2}{2}(\Delta^\sigma)^2 f + o(t^2),\; f + t\Delta^\sigma f + \frac{t^2}{2}(\Delta^\sigma)^2 f + o(t^2)\Big)$$

$$= \left(2t + 2Kt^2 + o(t^2)\right)\Big(\Gamma^\sigma(f,f) + t\Gamma^\sigma(\Delta^\sigma f, f) + t\Gamma^\sigma(f,\Delta^\sigma f) + t^2\Gamma^\sigma(\Delta^\sigma f, \Delta^\sigma f) + \frac{t^2}{2}\Gamma^\sigma(f, (\Delta^\sigma)^2 f) + \frac{t^2}{2}\Gamma^\sigma((\Delta^\sigma)^2 f, f) + o(t^2)\Big)$$

$$= 2t\Gamma^\sigma(f) + 2t^2\Gamma^\sigma(\Delta^\sigma f, f) + 2t^2\Gamma^\sigma(f,\Delta^\sigma f) + 2Kt^2\Gamma^\sigma(f,f) + o(t^2).$$

Therefore, we derive

$$\frac{t^2}{2}\Delta^2(f^2) - t^2(\Delta^\sigma f)^2 - t^2 f(\Delta^\sigma)^2 f + o(t^2)$$

$$\geq 2t^2\Gamma^\sigma(\Delta^\sigma f, f) + 2t^2\Gamma^\sigma(f,\Delta^\sigma f) + 2Kt^2\Gamma^\sigma(f) + o(t^2).$$

Dividing by $2t^2$ and letting $t \to 0$ yields

$$\frac{1}{4}\Delta^2(f^2) - \frac{1}{2}(\Delta^\sigma f)^2 - \frac{1}{2}f(\Delta^\sigma)^2 f \geq \Gamma^\sigma(\Delta^\sigma f, f) + \Gamma^\sigma(f,\Delta^\sigma f) + K\Gamma^\sigma(f), \quad \forall f.$$

Inserting $\Gamma^\sigma(\Delta^\sigma f, f) = \frac{1}{2}\left(\Delta(f\cdot\Delta^\sigma f) - f(\Delta^\sigma)^2 f - (\Delta^\sigma f)^2\right)$, we have

$$\frac{1}{4}\Delta^2(f^2) - \frac{1}{2}\Delta(f\cdot\Delta^\sigma f) + \cancel{\Gamma^\sigma(\Delta^\sigma f, f)} \geq \cancel{\Gamma^\sigma(\Delta^\sigma f, f)} + \Gamma^\sigma(f,\Delta^\sigma f) + K\Gamma^\sigma(f), \quad \forall f.$$

That is,

$$\frac{1}{2}\left(\frac{1}{2}\Delta^2(f^2) - \Delta(f\cdot\Delta^\sigma f)\right) = \frac{1}{2}\Delta\left(\frac{1}{2}\Delta(f^2) - f\cdot\Delta^\sigma f\right)$$

$$= \frac{1}{2}\Delta\Gamma^\sigma(f) \geq \Gamma^\sigma(f,\Delta^\sigma f) + K\Gamma^\sigma(f), \quad \forall f.$$

Hence

$$\underbrace{\frac{1}{2}\Delta\Gamma^\sigma(f) - \Gamma^\sigma(f,\Delta^\sigma f)}_{=\,\Gamma_2^\sigma(f)} \geq K\Gamma^\sigma(f), \quad \forall f.$$

$\square$

Lemma II.10.12: For any $f, g: V \to \mathbb{R}$, we have

(i) $\sum_{x\in V} \Gamma^\sigma(f,g)(x) d_x = -\langle f, \Delta^\sigma g\rangle_d = -\langle \Delta^\sigma f, g\rangle_d$

(ii) $|\Gamma^\sigma(f,g)(x)| \leq \sqrt{\Gamma^\sigma(f)(x)}\sqrt{\Gamma^\sigma(g)(x)}$.

Proof: (i) follows from $\Gamma^\sigma(f,g) = \frac{1}{2}(\Delta(fg) - f\Delta^\sigma g - \Delta^\sigma f \cdot g)$.

That is, $\text{LHS} = \underbrace{\frac{1}{2}\sum_{x\in V}\Delta(fg)(x)d_x}_{\circledast} - \frac{1}{2}\sum_{x\in V} f(x)\Delta^\sigma g(x) d_x - \frac{1}{2}\sum_{x\in V}\Delta^\sigma f(x) g(x) d_x$

$$\overset{\substack{\text{self-adjointness}\\ \text{of } \Delta \text{ and } \Delta^\sigma \\ \Delta 1 = 0}}{=} 0 - \langle f, \Delta^\sigma g\rangle = 0 - \langle \Delta^\sigma f, g\rangle.$$

(ii) follows from Cauchy–Schwarz. Indeed

$$\Gamma^\sigma(f,g)(x) = \frac{1}{2d_x}\sum_{y: y\sim x}(\sigma_{xy} f(y) - f(x))(\sigma_{xy} g(y) - g(x))$$

$$\leq \sqrt{\frac{1}{2d_x}\sum_{y:y\sim x}(\sigma_{xy}f(y) - f(x))^2}\sqrt{\frac{1}{2d_x}\sum_{y:y\sim x}(\sigma_{xy}g(y)-g(x))^2}. \quad \square$$

Lemma II.10.13: Let $\Gamma = (G, \sigma)$ satisfy $CD^\sigma(-K, \infty)$ for $K \geq 0$.

Then for any $f: V \to \mathbb{R}$, and any $t \geq 0$,

$$\|f - P_t^\sigma f\|_1 \leq \int_0^t \sqrt{\frac{K}{1-e^{-2Ks}}}\, ds \cdot \|\sqrt{\Gamma^\sigma(f)}\|_1. \quad (1)$$

Rmk: Here we use the notation for $1 \leq p \leq \infty$ that

$$\|f\|_p := \left(\sum_{x\in V} d_x |f(x)|^p\right)^{\frac{1}{p}}.$$

Proof: An immediate consequence of Theorem II.10.11 (iii)

is $\|\sqrt{P_t(f^2)}\|_\infty \geq \sqrt{\frac{1-e^{-2Kt}}{K}}\|\sqrt{\Gamma^\sigma(P_t^\sigma f)}\|_\infty$, $\forall f$. $(\circledast)$

We'll show (1) is a dual version of it in certain sense.

We set $g : V \to \mathbb{R}$ s.t.

$$g(x) = \begin{cases} 0 & \text{, if } f(x) - P_t^\sigma f(x) = 0 \\ \dfrac{f(x) - P_t^\sigma f(x)}{|f(x) - P_t^\sigma f(x)|} & \text{, otherwise.} \end{cases}$$

Then we calculate

$$\| f - P_t^\sigma f \|_1 = \langle f - P_t^\sigma f, \; g \rangle$$

$$= \langle -\int_0^t \frac{\partial}{\partial s} P_s^\sigma f ds, \; g \rangle$$

$$= - \int_0^t \langle \underbrace{\Delta^\sigma P_s^\sigma}_{= P_s^\sigma \Delta^\sigma} f, \; g \rangle \, ds$$

$$\overset{P_s^\sigma \text{ self-adj.}}{=} - \int_0^t \langle \Delta^\sigma f, \; P_s^\sigma g \rangle \, ds$$

$$\overset{\text{Lemma II.10.12 (i)}}{=} \int_0^t \sum_{x \in V} \Gamma^\sigma (f, P_s^\sigma g)(x) \, d_x \, ds$$

$$\overset{\text{Lemma II.10.12 (ii)}}{\leq} \int_0^t \sum_{x \in V} d_x \sqrt{\Gamma^\sigma(f)(x)} \sqrt{\Gamma^\sigma(P_s^\sigma g)} \, ds$$

$$\leq \int_0^t \| \sqrt{\Gamma^\sigma(f)} \|_1 \cdot \| \sqrt{\Gamma^\sigma(P_s^\sigma g)} \|_\infty \, ds$$

$$\overset{A}{\leq} \| \sqrt{\Gamma^\sigma(f)} \|_1 \int_0^t \sqrt{\frac{K}{1 - e^{-2Ks}}} \, \| \sqrt{P_s g^2} \|_\infty \, ds$$

$$\overset{P_s(g^2) \leq \|g^2\|_\infty = 1}{\leq} \| \sqrt{\Gamma^\sigma(f)} \|_1 \cdot \int_0^t \sqrt{\frac{K}{1 - e^{-2Ks}}} \, ds. \qquad \square$$

Rmk. When $K = 0$, we obtain:

$$\| f - P_t^\sigma f \|_1 \leq \sqrt{2t} \, \| \sqrt{\Gamma^\sigma(f)} \|_1 .$$