<u>Lemma II.10.14</u>. For any $f : V \to \mathbb{R}$, we have

$$\|\sqrt{\Gamma^\sigma(f)}\|_1 \leq \sqrt{2 d_{max}} \sum_{\{x,y\}} |\sigma_{xy} f(y) - f(x)|.$$

<u>Proof</u>: $\|\sqrt{\Gamma^\sigma(f)}\|_1 = \sum_{x \in V} d_x \sqrt{\frac{1}{2d_x} \sum_{y: y \sim x} (\sigma_{xy} f(y) - f(x))^2}$

$$= \sum_{x \in V} \sqrt{\frac{d_x}{2}} \sqrt{\sum_{y: y \sim x} (\sigma_{xy} f(y) - f(x))^2}$$

$$\leq \sqrt{\frac{d_{max}}{2}} \sum_{x \in V} \sqrt{\left(\sum_{y: y \sim x} |\sigma_{xy} f(y) - f(x)|\right)^2}$$

$$= \sqrt{\frac{d_{max}}{2}} \sum_{x \in V} \sum_{y: y \sim x} |\sigma_{xy} f(y) - f(x)|.$$

$$= \sqrt{2 d_{max}} \sum_{\{x,y\}} |\sigma_{xy} f(y) - f(x)|.$$

$\square$.

We're now well prepared for the proof of Theorem II.10.1.

<u>Proof of Thm II.10.1</u>: Let $\{S_i\}_{i=1}^k$ be an arbitrary nontrivial subpartition of V, i.e., $S_i \neq \phi$, $i = 1, \cdots, k$, $S_i \cap S_j = \phi$ $\forall i \neq j$.

For each S_i, let $\tau_i : S_i \to \{+1, -1\}$ be the function s.t.

$$\sum_{\{x,y\} \in E_{S_i}} |1 - \tau_i(x) \sigma_{xy} \tau_i(y)| = \min_{\tau : S_i \to \{+1,-1\}} \sum_{\{x,y\} \in E_{S_i}} |1 - \tau(x) \sigma_{xy} \tau(y)|$$

By abuse of notation, we also write $\tau_i : V \to \{+1, -1, 0\}$ ~~with~~ extended by $\tau_i|_{V \setminus S_i} \equiv 0$.

Lemma II.10.14 tells that for each i

$$\|\sqrt{\Gamma^\sigma(\tau_i)}\|_1 \leq \sqrt{2 d_{max}} \sum_{\{x,y\}} |\sigma_{xy} \tau_i(y) - \tau_i(x)|$$

$$= \sqrt{2 d_{max}} \left(\sum_{\{x,y\} \in E_{S_i}} |\sigma_{xy} \tau_i(y) - \tau_i(x)| + |\partial S_i| \right)$$

$$= \sqrt{2 d_{max}} \, \phi^\sigma(S_i) \cdot \mathrm{vol}(S_i) \qquad (*1)$$

On the other hand, Lemma II.10.13 yields for each i:

$$\begin{aligned}
\sqrt{2t}\,\|\sqrt{\Gamma^\sigma(\tau_i)}\|_1 &\geq \|\tau_i - P_t^\sigma \tau_i\|_1 \\
&= \sum_{x\in V} |\tau_i(x) - P_t^\sigma \tau_i(x)|\, d_x \\
&= \sum_{x\in V} |\tau_i(x) - P_t^\sigma \tau_i(x)|\, |\tau_i(x)|\, d_x \\
&= \sum_{x\in V} \left|\tau_i(x)^2 - \tau_i(x) P_t^\sigma \tau_i(x)\right| d_x \\
&\geq \sum_{x\in V} \left(\tau_i(x)^2 - \tau_i(x) P_t^\sigma \tau_i(x)\right) d_x \\
&= \|\tau_i\|_2^2 - \langle \tau_i, P_t^\sigma \tau_i\rangle \\
&= \|\tau_i\|_2^2 - \langle \tau_i, P_{t/2}^\sigma \cdot P_{t/2}^\sigma \tau_i\rangle \\
&= \|\tau_i\|_2^2 - \|P_{t/2}^\sigma \tau_i\|_2^2
\end{aligned}$$

Let $\{\psi_n\}_{n=1}^N$ be an orthonormal basis of $\ell^2(V)$ consisting of the eigenfunctions corresponding to $\{\lambda_n^\sigma\}_{n=1}^N$, respectively.

By the spectral decomposition (Prop. II.10.10), we have

$$\|P_{\frac{t}{2}}^\sigma \tau_i\|_2^2 = \sum_{n=1}^N e^{-t\lambda_n^\sigma} \langle \tau_i, \psi_n\rangle^2 .$$

Moreover, we have

$$\|\tau_i\|_2^2 = \sum_{n=1}^N \langle \tau_i, \psi_n\rangle^2 = \sum_{x\in V} |\tau_i(x)|^2 d_x = \mathrm{vol}(S_i) .$$

Therefore,

$$\sqrt{2t}\,\|\sqrt{\Gamma^\sigma(\tau_i)}\|_1 \geq \mathrm{vol}(S_i) - \sum_{n=1}^N e^{-t\lambda_n^\sigma} \langle \tau_i, \psi_n\rangle^2 . \qquad (*2)$$

Combing (*1) and (*2) yields for each S_i,

$$2\sqrt{t\, d_{\max}}\, \phi^\sigma(S_i) \geq 1 - \sum_{n=1}^N e^{-t\lambda_n^\sigma} \frac{\langle \tau_i, \psi_n\rangle^2}{\mathrm{vol}(S_i)} \qquad (*3)$$

In order to figure out an estimate for λ_k^σ, we estimate

$$2\sqrt{t\, d_{max}}\, \phi^\sigma(S_i) \geq 1 - \sum_{n=1}^{k-1} \frac{\langle \tau_i, \psi_n \rangle^2}{vol(S_i)} - e^{-t\lambda_k^\sigma} \cdot \underbrace{\sum_{n=k}^{N} \frac{\langle \tau_i, \psi_n \rangle^2}{vol(S_i)}}_{\leq 1}$$

$$\geq 1 - \underbrace{\sum_{n=1}^{k-1} \frac{\langle \tau_i, \psi_n \rangle^2}{vol(S_i)}}_{\leq 1} - e^{-t\lambda_k^\sigma}$$

Of course, we have

$$1 - \sum_{n=1}^{k-1} \frac{\langle \tau_i, \psi_n \rangle^2}{vol(S_i)} \geq 0 \qquad (***)$$

However, we hope for a positive lower bound of it.

Recall that, we only need an estimate for

$$\max_{1 \leq i \leq k} \phi^\sigma(S_i)$$

So we aim at finding at least one i_0 such that $(***)$ has a positive lower bound. For that purpose, we estimate the sum

$$\sum_{i=1}^{k} \sum_{n=1}^{k-1} \frac{\langle \tau_i, \psi_n \rangle^2}{vol(S_i)} = \sum_{n=1}^{k-1} \sum_{i=1}^{k} \left\langle \frac{\tau_i}{\sqrt{vol(S_i)}}, \psi_n \right\rangle^2$$

Observe that

$$\left\{ \frac{\tau_i}{\sqrt{vol(S_i)}}, i = 1, \cdots, k \right\}$$

is a ~~syste~~ sequence of orthonormal functions in $\ell^2(V)$.

Therefore, $\displaystyle\sum_{i=1}^{k} \left\langle \frac{\tau_i}{\sqrt{vol(S_i)}}, \psi_n \right\rangle^2 \leq \|\psi_n\|_2^2 = 1$

This leads to

$$\sum_{i=1}^{k} \sum_{n=1}^{k-1} \frac{\langle \tau_i, \psi_n \rangle^2}{vol(S_i)} \leq \sum_{n=1}^{k-1} \cdot 1 = k-1$$

There must exists at least one i_0 s.t.

$$\sum_{n=1}^{k-1} \frac{\langle \tau_{i_0}, \psi_n \rangle^2}{vol(S_{i_0})} \leq \frac{k-1}{k} = 1 - \frac{1}{k} \Rightarrow 1 - \sum_{n=1}^{k-1} \frac{\langle \tau_{i_0}, \psi_n \rangle^2}{vol(S_{i_0})} \geq \frac{1}{k}$$

We then arrive at

$$\boxed{2\sqrt{t\, d_{max}} \max_{1\le i\le k} \phi^\sigma(S_i) \ge \frac{1}{k} - e^{-t\lambda_k^\sigma}}.$$

For $\lambda_k^\sigma \neq 0$, we set $t = \dfrac{\log 2k}{\lambda_k^\sigma}$ to obtain

$$2\sqrt{\frac{\log 2k \cdot d_{max}}{\lambda_k^\sigma}} \cdot h_k^\sigma \ge \frac{1}{k} - e^{-\log 2k} = \frac{1}{k}.$$

Rearranging finishes the proof. $\square$.

In general, we have.

<u>Theorem II.10.15</u>. Let $\Gamma = (G, \sigma)$ satisfy $CD^\sigma(-K, \infty)$, $K \ge 0$.

Then we have

$$\lambda_k^\sigma \le \max\left\{ \frac{8k^2(\log k+1)}{(1-e^{-1})^2} d_{max} (h_k^\sigma)^2,\ 8k\sqrt{K\, d_{max}}\, h_k^\sigma + 2\log k\, K \right\}$$

<u>Proof</u>: We first observe the following elementary fact:

$$1 - e^{-u} \ge \frac{u}{2}, \quad \text{for } 0 \le u \le 1.$$

Therefore, for $0 \le t \le \frac{1}{2K}$, we have

$$\int_0^t \sqrt{\frac{K}{1-e^{-2Ks}}}\, ds \le \int_0^t \sqrt{\frac{K}{Ks}}\, ds = \int_0^t s^{-\frac{1}{2}} ds$$

$$= 2\sqrt{t}.$$

That is, for $0 \le t \le \frac{1}{2K}$, it holds due to Lemma II.10.13 that

$$\|f - P_t^\sigma f\|_1 \le 2\sqrt{t}\, \|\sqrt{\Gamma^\sigma(f)}\|_1.$$

Similarly as in the proof of Thm II.10.1, we obtain

for any nontrivial k-subpartition $\{S_i\}_{i=1}^{k}$, that

$$2\sqrt{2t\, d_{max}} \max_{1\le i\le k} \phi^{\sigma}(S_i) \ge \frac{1}{k} - e^{-t\lambda_k^{\sigma}}, \quad \text{for } 0\le t\le \frac{1}{2K}.$$

$$= \frac{1}{k}\left(1 - k e^{-t\lambda_k^{\sigma}}\right)$$

$$= \frac{1}{k}\left(1 - e^{-t\lambda_k^{\sigma} + \log k}\right) \qquad (\ast\ast)$$

<1> If $\lambda_k^{\sigma} \ge 2K(\log k + 1)$, set $t = \dfrac{\log k + 1}{\lambda_k^{\sigma}} \le \dfrac{1}{2K}$,

we derive

$$2\sqrt{\frac{2(\log k+1)d_{max}}{\lambda_k^{\sigma}}}\, h_k^{\sigma} \ge \frac{1}{k}(1-e^{-1}),$$

that is, $\sqrt{\lambda_k^{\sigma}} \le \dfrac{k}{1-e^{-1}} \cdot 2\sqrt{2(\log k+1)d_{max}}\, h_k^{\sigma}$.

<2> If, otherwise, $\lambda_k^{\sigma} < 2K(\log k+1)$, set $t = \dfrac{1}{2K}$.

Then $t\lambda_k^{\sigma} < \log k + 1 \Rightarrow t\lambda_k^{\sigma} - \log k < 1$.

We derive from $(\ast\ast)$ that

$$2\sqrt{\frac{2d_{max}}{2K}}\, h_k^{\sigma} \ge \frac{1}{k}\left(1 - e^{-t\lambda_k^{\sigma} + \log k}\right)$$

$$\ge \frac{1}{k} \cdot \frac{\frac{\lambda_k^{\sigma}}{2K} - \log k}{2}$$

$$\Rightarrow \frac{\lambda_k^{\sigma}}{2K} \le 2k \cdot 2\sqrt{\frac{d_{max}}{K}}\, h_k^{\sigma} + \log k$$

$$\Rightarrow \lambda_k^{\sigma} \le 8k\sqrt{K\, d_{max}}\, h_k^{\sigma} + 2\log k\, K. \qquad \square$$

<u>Rmk</u>. For the case $k=1$, we have

$$\lambda_1^\sigma \le \max\left\{ \frac{8k^2}{(1-e^{-1})^2} d_{max} (h_k^\sigma)^2,\ 8k\sqrt{K d_{max}}\, h_k^\sigma \right\}$$

where K is s.t. $CD^\sigma(-K,\infty)$ holds.

Combining Thm II.10.1 with the improved Cheeger ineq from II.9, we show the following eigenvalue ratio estimates:

<u>Theorem ~~II.10.1b~~</u>: For any unbalanced $\Gamma = (G,\sigma)$ satisfying $CD^\sigma(0,\infty)$ and any $k \in \{1,\dots,N=|V|\}$, there exists a constant C s.t.

$$\lambda_k^\sigma \le C\, d_{max}\, k^2 \lambda_1^\sigma .$$

We can restate in a unified way:

$\lambda_k^\sigma \le C\, d_{max}\, k^2 \lambda^\sigma$

λ^σ is the first nonzero eigenvalue.

For any balanced $\Gamma = (G,\sigma)$ satisfying $CD^\sigma(0,\infty)$ and any $k \in \{1,\dots,N=|V|\}$, $\exists\, C$ s.t.

$$\lambda_k^\sigma \le C\, d_{max}\, k^2 \lambda_2^\sigma .$$

<u>Proof</u>: By Thm II.10.1, we have

$$\lambda_k^\sigma \le C\, d_{max} (h_1^\sigma)^2$$

$$\lambda_2^\sigma \le \tilde{C}\, d_{max} (h_2^\sigma)^2 .$$

Combining with the improved Cheeger ineq:

$$h_1^\sigma \le \bar{C} k \frac{\lambda_1^\sigma}{\sqrt{\lambda_k^\sigma}}$$

$$h_2^\sigma \le \bar{C}' k \frac{\lambda_2^\sigma}{\sqrt{\lambda_k^\sigma}}$$

yields

$$\sqrt{\lambda_1^\sigma} \le C\sqrt{d_{max}}\, k \frac{\lambda_1^\sigma}{\sqrt{\lambda_k^\sigma}} \quad \text{and} \quad \sqrt{\lambda_2^\sigma} \le C\sqrt{d_{max}}\, k \frac{\lambda_2^\sigma}{\sqrt{\lambda_k^\sigma}} .$$

If $\lambda_1^\sigma \neq 0$, i.e. σ is non balanced, we obtain

$$\lambda_k^\sigma \leq C\, d_{max}\, k^2 \lambda_1^\sigma$$

If σ is balanced, then we have

$$\lambda_k^\sigma \leq C\, d_{max}\, k^2 \lambda_2^\sigma. \qquad \square$$

<u>Rmk</u>: Consider the cycle graphs, the order "k^2" is optimal.

This further implies the optimality of the order of k in the improved Cheeger-ineq.

<u>II.11.</u> Lichnerowicz estimate

In this section, we briefly discuss the following estimate:

<u>Theorem II.11.1</u> (Lichnerowicz type estimate). Let $\Gamma = (G, \sigma)$ satisfy $CD^\sigma(K, \infty)$, Then for any nonzero eigenvalue λ^σ of Δ^σ:

$$\lambda^\sigma \geq K.$$

<u>Proof</u>: $CD^\sigma(K,\infty) \Rightarrow \Gamma_2^\sigma(f)(x) \geq K\Gamma^\sigma(f)(x)$, $\forall f: V \to \mathbb{R}$

Summing over vertices w.r.t. degree, we have

$$\sum_{x\in V} \Gamma_2^\sigma(f)(x)\, d_x \geq K \sum_{x\in V} \Gamma^\sigma(f)(x)\, d_x$$

where $LHS = -\sum_{x\in V} d_x\, \Gamma^\sigma(f, \Delta^\sigma f)(x)$

$RHS = -K\langle f, \Delta^\sigma f\rangle$

<u>Cor</u>: Let $\Gamma = (G,\sigma)$ satisfy $CD^\sigma(K,\infty)$. If σ is not balanced, we have

$$K \leq 2h_1^\sigma.$$

If σ is balanced,

$$K \leq 2h_2^\sigma.$$

Set f be the eigenfct w.r.t. λ^σ. Then

$$\lambda^\sigma \underbrace{\sum_{x\in V} d_x\, \Gamma^\sigma(f,f)(x)}_{=\langle f, -\Delta^\sigma f\rangle = \lambda^\sigma\langle f, f\rangle} \geq \lambda^\sigma K \langle f, f\rangle \Rightarrow (\lambda^\sigma)^2 \geq \lambda^\sigma (K). \qquad \square$$

More comments about the curvature.

The optimal K s.t. $\Gamma_2^\sigma(f)(x) \geq K\Gamma^\sigma(f)(x)$ at a given vertex can be calculate via solving a semidefinite programming problem:

Observe that there exists two symmetric $N \times N$-matrices, $N=|V|$, $\Gamma^\sigma(x)$ and $\Gamma_2^\sigma(x)$ s.t. for any $f, g: V \longrightarrow \mathbb{R}$

$$\Gamma^\sigma(f,g)(x) = \vec{f}^T \Gamma^\sigma(x) \vec{g}$$

and $\Gamma_2^\sigma(f,g)(x) = \vec{f}^T \Gamma_2^\sigma(x) \vec{g}$.

~~The optimal~~ Finding optimal K in $CD^\sigma(K,\infty)$ is equivalent to solving:

$$\max \quad K$$
$$\text{subject to} \quad \underbrace{\Gamma_2^\sigma(x) \geq K\Gamma^\sigma(x)}_{\text{i.e. } \Gamma_2^\sigma(x) - K\Gamma^\sigma(x) \text{ is semidefinite.}}$$

Denote the solution by $K_\infty^\sigma(x)$

One can ~~calculate~~ derive explicit expressions of the local matrices $\Gamma^\sigma(x), \Gamma_2^\sigma(x)$. We refer for details to [LMP19, §3.4].

<u>Example</u>: (Cycles C_N)

- $N=3$, $(\triangle, \sigma)$, $K_\infty^\sigma(x) = \begin{cases} \frac{5}{4}, & \text{if } \sigma \text{ is balanced} \\ \frac{1}{4}, & \text{if } \sigma \text{ is unbalanced.} \end{cases}$

- $N=4$, $(\square, \sigma)$, $K_\infty^\sigma(x) = \begin{cases} 1, & \text{if } \sigma \text{ is balanced.} \\ 0, & \text{if } \sigma \text{ is unbalanced.} \end{cases}$

- $N \geq 5$, $K_\infty^\sigma(x) = 0$.

Recall from Lecture 4 (pp. 31), when σ is balanced.

All Laplacian eigenvalues of the N-cycle are

$$1-\cos\frac{2\pi j}{N}, \quad j=0,1,\cdots,N-1.$$

If we write them in ~~increasing~~ non-decreasing order, we have

$$\lambda_k = 1-\cos\left(\frac{2\pi}{N}\left\lfloor\frac{k}{2}\right\rfloor\right), \quad k=1,2,\cdots,N.$$

Observe that we have

$$\lim_{N\to\infty}\frac{\lambda_k}{\lambda_2} = \left\lfloor\frac{k}{2}\right\rfloor^2$$

It is straightforward to check that

$$2-\lambda_{N-k+1} = \begin{cases} 1-\cos\frac{(k-1)\pi}{N}, & \text{if } N-k+1 \text{ is even} \\ 1-\cos\frac{k\pi}{N}, & \text{if } N-k+1 \text{ is odd.} \end{cases}$$

Recall if $\sigma_0 \equiv -1$, (C_N, σ_0) is balanced iff N even

For odd N, (C_N, σ_0) is unbalanced, and

$$\lambda_k^\sigma = 2-\lambda_{N-k+1}, \quad k=1,2,\cdots,N.$$

Observe that

$$\lim_{N\to\infty}\frac{\lambda_k^\sigma}{\lambda_1^\sigma} = \lim_{N\to\infty}\frac{\lambda_k^\sigma}{1-\cos\frac{\pi}{N}} \approx k^2.$$