the pointwise limit of f_r.

For every closed interval $[0,T]$, we see ~~$f_r' |_{[0,1]}$ is~~

$$f_r'(t) = \frac{\partial}{\partial t} p_t^r(x,y) = \Delta_{B_r} p_t^r(x,y) \leq 2$$

~~is~~ has an upper bound indep. of r.

This tells that f is continuous on $[0,T]$.

That is: (i) $f_r \leq f_{r+1}$ on $[0,T]$

(ii) f_r continuous on $[0,T]$

(iii) $f_r \to f$ pointwisely, f is continuous on $[0,T]$

By Dini Theorem, we have $\{f_r\}_{r=1}^{\infty}$ converges <u>uniformly</u> to f.

We hope to check whether the limit $f(t) = p_t^{\Phi}(x,y)$ provides a "fundamental solution" to the heat equation. The question is whether the derivative $\frac{\partial}{\partial t} p_t^{\Phi}(x,y) = f'(t)$ exists, and if yes, whether it equals $\Delta p_t(x,y) = \frac{1}{d_x} \sum_{z \sim x} (p_t(z,y) - p_t(x,y))$?

We need the following result:

<u>Theorem</u> III.2.8: Suppose $\{f_r\}_{r=1}^{\infty}$ is a sequence of functions, <u>differentiable</u> on $[a,b]$ and such that $\{f_r(x_0)\}_{r=1}^{\infty}$ converges for some $x_0 \in [a,b]$. If $\{f_r'\}_{r=1}^{\infty}$ <u>converges uniformly</u> on $[a,b]$, then $\{f_r\}_{r=1}^{\infty}$ converges uniformly on $[a,b]$ to a function f and for all $x \in [a,b]$,

$$f'(x) = \lim_{r \to \infty} f_r'(x).$$

~~For~~ In general, That $\{f_r\}$ converges uniformly does not implies that $\{f_r'\}$ converges uniformly. However, in our case, our functions satisfy "equations":

$$f_r'(t) = \frac{\partial}{\partial t} p_t^r(x,y) = \Delta p_t^r(x,y) \underset{\uparrow}{=} \frac{1}{d_x}\sum_{z\sim x}\left(p_t^r(z,y) - p_t^r(x,y)\right)$$

for r large enough.

Recall $\forall\, x,y$, $t \mapsto p_t^r(x,y)$ converges uniformly as $r \to \infty$.

Therefore $f_r'(t) = \frac{\partial}{\partial t} p_t^r(x,y)$ converges uniformly.

By Thm III.2.8, we have

$$\cancel{f_r'(t) = \frac{\partial}{\partial t} p} \quad \underset{\substack{\| \\ \frac{\partial}{\partial t} p_t(x,y)}}{f'(t)} = \lim_{r\to\infty} f_r'(t) \overset{\substack{\text{equation} \\ \downarrow}}{=} \lim_{r\to\infty} \Delta p_t^r(x,y)$$

$$= \Delta p_t(x,y).$$

That is, $p_t(x,y)$ is a solution of the heat equation in either x or y.

Moreover, $p_0(x,y) = \lim_{r\to\infty} p_0^r(x,y) = \frac{\delta_y}{d_y}(x) = \frac{\delta_x}{d_x}(y)$.

So $p_t(x,y)$ is a fundamental solution !!

In ~~dead~~ fact, we observe that

$$\frac{\partial}{\partial t} p_t^r(x,y) = \Delta p_t^r(x,y) = \frac{1}{d_x}\sum_{z\sim x}\left(p_t^r(z,y) - p_t^r(x,y)\right)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} p_t^r(x,y) = \Delta \frac{\partial}{\partial t} p_t^r(x,y) = \frac{1}{d_x}\sum_{z\sim x}\Big(\underset{\substack{\uparrow \\ \text{uniformly converge}}}{\frac{\partial}{\partial t} p_t^r(z,y) - \frac{\partial}{\partial t} p_t^r(x,y)}\Big)$$

So Thm III.2.8 implies that

$$\frac{\partial^2}{\partial t^2} p_t(x,y) = \lim_{r\to\infty} \frac{\partial^2}{\partial t^2} p_t^r(x,y) = \Delta \frac{\partial}{\partial t} p_t(x,y).$$

Applying this procedure iteratively, we show that $p_t(x,y)$ is C^∞ in t.

<u>Lemma III.2.9</u> p is independent of the exhaustion used to define it.

<u>Proof</u>: Consider another exhaustion of G: $\{D_i\}_{i=1}^{\infty}$, that is, each D_i is a finite and connected sub-graph, $D_i \subset D_{i+1}$ for all i, and $G = \bigcup_{i=1}^{\infty} D_i$. Let $q_t^{D_i}(x,y)$ denote the Dirichlet heat kernels for this exhaustion, and say that $q_t^{D_i}(x,y) \longrightarrow q_t(x,y)$ as $i \to \infty$.

Then for each D_i, there exists r large enough s.t. $D_i \subset B_r(x_0)$. Recall $q_t^{D_i}(x,y)$ vanishes on $\mathrm{bdy}\, D_i$. By Maximum principle, we derive

$$q_t^{D_i}(x,y) \le p_t^r(x,y)$$

By Lemma III.2.7, we have by letting $r \to \infty$.

$$q_t^{D_i}(x,y) \le p_t(x,y).$$

Letting $i \to \infty$ yields

$$q_t(x,y) \le p_t(x,y).$$

Similarly, we derive

$$p_t(x,y) \le q_t(x,y).$$

Therefore, $p_t(x,y) = q_t(x,y)$. $\square$

We also want to mention the following properties of p:

① $p_t(x,y) = p_t(y,x)$, $p_t(x,y) > 0$, $\forall x,y \in V, \forall t > 0$

② $p_{s+t}(x,y) = \sum_{z \in V} p_s(x,z)\, p_t(z,y)\, d_z$

③ $\sum_{y \in V} p_t(x,y)\, d_y \le 1$, $\forall t \ge 0$, $\forall x \in V$

They follows direct from corresp. prop. of p_t^r.

<u>Lemma III.2.8</u>. p is the smallest non-negative function that satisfies

$$\begin{cases} \Delta q_t(x,y) = \frac{\partial}{\partial t} q_t(x,y) \\ q_0(x,y) = \frac{\delta_y}{d_y}(x). \quad \forall x,y \in V \end{cases}$$

Δ is either in x or y

Proof: This again follows from maximum principle.

Say $q_t(x,y)$ is another non-negative function that satisfies the equations. Then both q and p^r satisfies the heat equation on $\mathrm{int}B_r \times [0,T]$. Since p^r vanishes on bdy B_r while $q \geq 0$, we derive from maximum principle that

$$q_t - p_t^r \geq 0 \quad \text{on } B_r \times [0,T].$$

Therefore, $q_t(x,y) - p_t^r(x,y) \geq 0$, $\forall x,y$, $\forall t>0$.

Letting $r \to \infty$ yields $q_t(x,y) \geq p_t(x,y)$. $\square$.

Two approaches coincide

Theorem III.2.9: For any $f \in \ell^2(V)$, we have

$$e^{t\Delta} f(x) = \sum_{y\in V} p_t(x,y) f(y)\, d_y, \quad \forall x \in V.$$

Rmk: That is, $p_t(x,y) = e^{t\Delta}\left(\frac{\delta_y}{d_y}\right)(x) = \left\langle e^{t\Delta}\left(\frac{\delta_y}{d_y}\right), \frac{\delta_x}{d_x}\right\rangle$.

The kernel defined via ~~the~~ $e^{t\Delta}$ coincides with the kernel constructed from exhaustion.

Proof: Let us denote the operator $\bar{P}_t : \ell^2(V) \to \ell^2(V)$ s.t.

$$\bar{P}_t f(x) := \sum_{y \in V} p_t(x,y) f(y)\, d_y.$$

Of course, we have to first justify that $\bar{P}_t f \in \ell^2(V)$ for $f \in \ell^2(V)$.

First, for any $f_0 \in C_0(V)$, a function with finite support, choose r large enough s.t. $\mathrm{supp}(f_0) \subset B_r(x_0)$. Then we have

$$\cancel{\|P_t f_0\|_{\ell^2(V)}} \quad \sum_{y\in V} p_t^r(x,y) f_0(y)\, dy = \sum_{y\in B_r} p_t^r(x,y) f_0(y)\, dy$$

$$= e^{t\Delta_{B_r}} f_0(x)$$

By spectral decomposition,

$$\|e^{t\Delta_{B_r}} f_0\|_{\ell^2(V)} = \Big\|\sum_{n=1}^{|\mathrm{int} B_r|} e^{-t\lambda_n^r} \langle \phi_n^r, f_0\rangle \phi_n^r\Big\|_{\ell^2} = \sum_{n=1}^{|\mathrm{int} B_r|} e^{-2t\lambda_n^r} \langle \phi_n^r, f_0\rangle^2$$

$$\leq \sum_{n=1}^{|\mathrm{int} B_r|} \langle \phi_n^r, f_0\rangle^2 = \|f_0\|_{\ell^2(V)}$$

That is,

$$\sum_{x\in V}\Big(\sum_{y\in V} p_t^r(x,y) f_0(y)\, dy\Big)^2 dx \leq \|f_0\|_{\ell^2(V)}^2 .$$

↑ This is a finite sum.

By Fatou's lemma, we derive

$$\sum_{x\in V}\Big(\sum_{y\in V} p_t(x,y) f_0(y)\, dy\Big)^2 dx = \sum_{x\in V} \lim_{r\to\infty}\Big(\sum_{y\in V} p_t^r(x,y) f_0(y)\, dy\Big)^2 dx$$

$$\leq \lim_{r\to\infty} \sum_{x\in V}\Big(\sum_{y\in V} p_t^r(x,y) f_0(y)\, dy\Big)^2 dx$$

$$\leq \|f_0\|_{\ell^2(V)}^2$$

That is, $\|\bar{P}_t f_0\|_{\ell^2(V)} \leq \|f_0\|_{\ell^2(V)}^2$.

Now ~~for~~ any $f\in \ell^2(V)$, take a sequence $f_\varepsilon \in C_0(V)$, s.t. $\lim_{\varepsilon\to 0}\|f-f_\varepsilon\|_{\ell^2(V)} = 0$. We can ensure that $|f_\varepsilon| \leq |f|$.

Then $$\|\bar{P}_t f\|_{\ell^2(V)}^2 = \sum_{x\in V}\Big(\sum_{y\in V}\lim_{\varepsilon\to 0} p_t(x,y) f_\varepsilon(y)\, dy\Big)^2 dx$$

Since $|p_t(x,y) f_\varepsilon(y)| \leq p_t(x,y)|f|(y)$

where $$\sum_{y\in V} p_t(x,y)|f(y)|\, dy \leq \Big(\sum_{y\in V} p_t(x,y)^2 dy\Big)^{1/2}\Big(\sum_{y\in V} f(y)^2 dy\Big)^{1/2}$$
$$\leq \|f\|_{\ell^2(V)} < \infty .$$

We derive from dominated convergence theorem that

$$\sum_{y\in V} \lim_{\varepsilon\to 0} p_t(x,y) f_\varepsilon(y)\, dy = \lim_{\varepsilon\to 0} \sum_{y\in V} p_t(x,y) f_\varepsilon(y)\, dy.$$

Now, Fatou's lemma yields

$$\|\overline{P}_t f\|^2_{\ell^2(V)} = \sum_{x\in V} \Big(\sum_{y\in V} \lim_{\varepsilon\to 0} p_t(x,y) f_\varepsilon(y)\, dy\Big)^2 dx$$

$$= \sum_{x\in V} \lim_{\varepsilon\to 0} \Big(\sum_{y\in V} p_t(x,y) f_\varepsilon(y)\, dy\Big)^2 dx$$

$$\overset{\text{Fatou}}{\leq} \lim_{\varepsilon\to 0} \|\overline{P}_t f_\varepsilon\|^2_{\ell^2(V)} = \lim_{\varepsilon\to 0} \|f_\varepsilon\|^2_{\ell^2(V)} = \|f\|^2_{\ell^2(V)}.$$

That is, $\overline{P}_t f \in \ell^2(V)$ for any $f \in \ell^2(V)$.

Moreover, we infact ~~pro~~ show that $\|\overline{P}_t\| \leq 1$.

That is $\overline{P}_t$ is a contraction.

Next, we consider for any $f_0 \in C_0(V)$, that

$$u(x,t) := \left(\overline{P}_t - e^{t\Delta}\right) f_0(x).$$

Then

$$u(x,0) = \overline{P}_0 f_0(x) - f_0(x)$$

$$= \sum_{y\in V} p_0(x,y) f_0(y)\, dy - f_0(x)$$

$$= \sum_{y\in V} \frac{\delta_x}{d_x}(y) f_0(y)\, dy - f_0(x) = f_0(x) - f_0(x) = 0.$$

We calculate

$$\sum_{x\in V} u^2(x,t)\, dx = \sum_{x\in V} \int_0^t \frac{\partial}{\partial \tau} u^2(x,\tau)\, d\tau\, dx$$

$$= 2\sum_{x\in V} \int_0^t u(x,\tau) \frac{\partial}{\partial \tau} u(x,\tau)\, d\tau\, dx$$

$$= 2\sum_{x\in V} \int_0^t u(x,\tau)\, \Delta u(x,\tau)\, d\tau\, dx. \qquad (*)$$

Notice that $u \in \ell^2(V,x)$, and $\Delta u \in \ell^2(V,x)$ (since $\Delta : \ell^2(V) \to \ell^2(V)$ is bounded).

Observe that

$$\int_0^t \sum_{x\in V} |u(x,\tau)\, \Delta u(x,\tau)| dx\, d\tau$$

$$\leq \int_0^t \Big(\sum_{x\in V} u(x,\tau)^2 dx\Big)^{\frac{1}{2}} \Big(\sum_{x\in V} \Delta u(x,\tau)^2 dx\Big)^{\frac{1}{2}} d\tau$$

$$= \int_0^t \|u(\cdot,\tau)\|_{\ell^2(V)} \|\Delta u(\cdot,\tau)\|_{\ell^2(V)} d\tau \,.\,.$$

$$\|u(\cdot,\tau)\|_{\ell^2(V)} = \|\bar{P}_\tau f_0 - e^{\tau\Delta} f_0\|_{\ell^2(V)} \leq \|\bar{P}_\tau f_0\|_{\ell^2(V)} + \|e^{\tau\Delta} f_0\|_{\ell^2(V)}$$
$$\leq 2\|f_0\|_{\ell^2(V)} \oplus$$

and hence

$$\|\Delta u(\cdot\,\tau)\|_{\ell^2(V)} \leq 2\|u(\cdot,\tau)\|_{\ell^2(V)} \leq 4\|f_0\|_{\ell^2(V)}.$$

Therefore. $\int_0^t \sum_{x\in V} |u(x,\tau)\, \Delta u(x,\tau)| dx\, d\tau$

$$\leq 8\|f_0\|_{\ell^2(V)} \cdot t \,.\, < \infty.$$

We can then apply Fubini theorem to (*) to obtain

$$\sum_{x\in V} u^2(x,t)\, dx = 2\sum_{x\in V} \int_0^t u(x,\tau)\, \Delta u(x,\tau)\, d\tau\, dx$$

$$= 2\int_0^t \sum_{x\in V} u(x,\tau) \Delta u(x,\tau)\, dx\, d\tau$$

$$\overset{\text{Prop III.2.2 pp.183}}{=} -2\int_0^t \frac{1}{2} \sum_{x\in V} \sum_{y: y\sim x} (u(y,\tau) - u(x,\tau))^2 d\tau$$

$$\leq 0.$$

Therefore $u(x,t) = 0,\ \forall x\in V,\ \forall t\geq 0.$ $\Rightarrow \bar{P}_t = e^{t\Delta}$ on $C_0(V)$.

Since both P_t and $e^{t\Delta}$ are bounded, we have $P_t = e^{t\Delta}$ on $\ell^2(V)$ □.

<u>Corollary III.2.10</u>: The heat semi-group $e^{t\Delta}$ is positive, i.e.

$$e^{t\Delta} f \geq 0 \text{ if } f \geq 0.$$