Observe that

$$\int_0^t \sum_{x\in V} |u(x,\tau)\,\Delta u(x,\tau)|\,dx\,d\tau$$

$$\leq \int_0^t \Big(\sum_{x\in V} u(x,\tau)^2 dx\Big)^{1/2}\Big(\sum_{x\in V} \Delta u(x,\tau)^2 dx\Big)^{1/2} d\tau$$

$$= \int_0^t \|u(\cdot,\tau)\|_{\ell^2(V)} \|\Delta u(\cdot,\tau)\|_{\ell^2(V)}\, d\tau \,..$$

$$\|u(\cdot,\tau)\|_{\ell^2(V)} = \|\bar{P}_\tau f_0 - e^{\tau\Delta} f_0\|_{\ell^2(V)} \leq \|\bar{P}_\tau f_0\|_{\ell^2(V)} + \|e^{\tau\Delta} f_0\|_{\ell^2(V)}$$
$$\leq 2\|f_0\|_{\ell^2(V)} \oplus$$

and hence

$$\|\Delta u(\cdot\,\tau)\|_{\ell^2(V)} \leq 2\|u(\cdot,\tau)\|_{\ell^2(V)} \leq 4\|f_0\|_{\ell^2(V)}.$$

Therefore. $\int_0^t \sum_{x\in V} |u(x,\tau)\,\Delta u(x,\tau)|\,dx\,d\tau$

$$\leq 8\|f_0\|_{\ell^2(V)} \cdot t\,. < \infty.$$

We can then apply Fubini theorem to $(*)$ to obtain

$$\sum_{x\in V} u^2(x,t)\,dx = 2\sum_{x\in V}\int_0^t u(x,\tau)\,\Delta u(x,\tau)\,d\tau\,dx$$

$$= 2\int_0^t \sum_{x\in V} u(x,\tau)\Delta u(x,\tau)\,dx\,d\tau$$

$$\overset{\text{Prop III.2.2 pp.183}}{=} -2\int_0^t \frac{1}{2}\sum_{x\in V}\sum_{y: y\sim x} (u(y,\tau)-u(x,\tau))^2\,d\tau$$

$$\leq 0.$$

Therefore $u(x,t) = 0$, $\forall x\in V$, $\forall t\geq 0$. $\Rightarrow \bar{P}_t = e^{t\Delta}$ on $C_0(V)$.

Since both P_t and $e^{t\Delta}$ are bounded, we have $P_t = e^{t\Delta}$ on $\ell^2(V)$ □.

<u>Corollary III.2.10</u>: The heat semi-group $e^{t\Delta}$ is positive, i.e.

$$e^{t\Delta} f \geq 0 \text{ if } f \geq 0.$$

<u>Corollary III.2.11</u>: The heat equation has a solution for any bounded initial function f. (Proof: $P_t f(x) = \sum_{y\in V} P_t(x,y) f(y)\,d_y$ converges).

(III.3) Stochastic completeness

Let $G=(V,E)$ be a locally finite infinite graph. Let $\mathbb{1}$ be a function taking value 1 at each vertex. Then

$$P_t \mathbb{1}(x) = \sum_{y \in V} p_t(x,y)\, d_y \le 1$$

Definition III.3.1. A graph G is called stochastically incomplete if for some vertex x_0 of G and some $t_0>0$ that

$$P_{t_0} \mathbb{1}(x_0) = \sum_{y \in V} p_t(x_0,y)\, d_y < 1.$$

Theorem III.3.2 TFAE:

(1) For some $t_0>0$, some $x_0 \in V$, $P_{t_0}\mathbb{1}(x_0) < 1$.

(1a) For all $t>0$, all $x \in V$, $P_t \mathbb{1}(x) < 1$.

(2) There exists a positive bounded function f on G s.t. $\Delta f + \lambda f = 0$ for any $\lambda < 0$.

(2a) There exists a positive bounded function f on G s.t. $\Delta f + \lambda f \ge 0$ for any $\lambda < 0$.

(3) There exists a nonzero, bounded solution to

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \Delta u(x,t), & \text{for all } x \in V, \text{ all } t>0 \\ u(x,0) = 0, & \text{for all } x \in V. \end{cases}$$

Proof: Our plan:

$$\begin{tikzcd}
(1a) \arrow[r, Rightarrow] & (2) \arrow[r, Rightarrow] & (3) \arrow[dll, Rightarrow] \\
 & (2a) \arrow[u, Rightarrow] & \\
(1) \arrow[uu, Rightarrow] & &
\end{tikzcd}$$

Note that $(1a) \Rightarrow (1)$ and $(2) \Rightarrow (2a)$ are trivial.

(1) $\Rightarrow$ (1a). This follows from maximal principle and semigroup property: Suppose $\exists\, x_0 \in V$ and $t_0 > 0$ s.t.

$$P_{t_0} \mathbb{1}(x_0) = 1$$

then $P_t \mathbb{1}$ attains its maximum at (x_0, t_0). By the maximum principle Lemma III. 2.4.(1) (pp. 185), we obtain

$$P_{t_0} \mathbb{1}(x) = 1 \quad \text{for all } x \in V. \quad (\ast)$$

Next, we show this is still true for any t by semigroup properties of P_t. If $s < t_0$, then we have

$$P_{t_0} \mathbb{1} = P_{t_0 - s}(P_s \mathbb{1}) \leq P_{t_0 - s} \mathbb{1} \leq 1.$$

By $(\ast)$, the inequalities above are equalities. In particular,

$$P_s \mathbb{1} = \mathbb{1} \qquad \text{for all } s < t_0.$$

If, otherwise, $s > t_0$, then $\exists\, k \in \mathbb{Z}$ s.t. $s < k t_0$. On the other hand, we have by semigroup property:

$$P_{k t_0} \mathbb{1} = \underbrace{P_{t_0} \cdots P_{t_0}}_{k \text{ times}} \mathbb{1} = \mathbb{1}.$$

Therefore, we have $P_s \mathbb{1} = \mathbb{1}$. $\square$

(1a) $\Rightarrow$ (2) We call a function f s.t. $\Delta f + \lambda f = 0$ a <u>λ-harmonic</u>.

We construct such a λ-harmonic for any $\lambda < 0$ explicitly from $u(x,t) := P_t \mathbb{1}(x)$. By assumption,

$$u(x,t) < 1 \quad \forall\, x \in V, \ \forall\, t > 0.$$

Let us first define

$$w(x) := \int_0^\infty e^{\lambda t} u(x,t)\, dt.$$

Then

$$0 < w < \int_0^\infty e^{\lambda t}\, dt = \frac{1}{\lambda} e^{\lambda t}\Big|_0^\infty = -\frac{1}{\lambda}. \quad (**)$$

Furthermore, we have

$$\Delta w = \Delta \int_0^\infty e^{\lambda t} u(\cdot,t)\, dt = \int_0^\infty e^{\lambda t} \Delta u(\cdot,t)\, dt$$

$$\overset{\text{heat}}{\underset{\text{eq.}}{=}} \int_0^\infty e^{\lambda t} \frac{\partial}{\partial t} u(\cdot,t)\, dt$$

$$\underset{\text{Integration by parts}}{=} e^{\lambda t} \underbrace{u(\cdot,t)}_{<1}\Big|_0^\infty - \int_0^\infty \frac{\partial}{\partial t} e^{\lambda t} \cdot u(\cdot,t)\, dt$$

$$= -1 - \lambda w(\cdot).$$

Define $f_\lambda := 1 + \lambda w$, then we have

$$\begin{aligned} \Delta f_\lambda + \lambda f_\lambda &= \Delta(1+\lambda w) + \lambda(1+\lambda w) \\ &= \lambda \Delta w + \lambda(1+\lambda w) \\ &= -\lambda(1+\lambda w) + \lambda(1+\lambda w) \\ &= 0. \end{aligned}$$

That is, f_λ is a λ-harmonic.

By (**), we have

$$0 = 1 + \lambda\cdot\left(-\frac{1}{\lambda}\right) < f_\lambda = 1 + \underbrace{\lambda}_{<0} \underbrace{w}_{(0,-\frac{1}{\lambda})} < 1.$$

That is, f_λ is positive and bounded. □

(3) ⇒ (1): Let T be any positive number. By (3), there exists a non-zero, bounded function $u(x,t)$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), & \forall\ (x,t) \in V \times (0,T) \\ u(x,0) = 0, & \forall\ x \in V. \end{cases}$$

Next, we construct a solution of the heat equation, which can be compared with the solution $P_t \mathbb{1}$ via maximum principle. Apparently, $u(x,t)$ is not such a solution we're looking ~~for~~ for since it has different initial data from $P_t \mathbb{1}$.

By rescaling, we may assume that $|u(x,t)| < 1$ for all $(x,t) \in V \times (0,T)$. and that $\exists\ x_0 \in V$ and $t_0 \in (0,T)$ s.t. $u(x_0,t_0) > 0$. Then define

$$w(x,t) := 1 - u(x,t). \quad \forall\ (x,t) \in V \times (0,T)$$

Observe that w is bounded, positive and satisfies

$$\begin{cases} \frac{\partial}{\partial t} w(x,t) = \Delta w(x,t), & \forall\ (x,t) \in V \times (0,T) \\ w(x,0) = 1, & \forall\ x \in V. \end{cases}$$

Furthermore, at (x_0,t_0), we have $w(x_0,t_0) = 1 - u(x_0,t_0) < 1$.

For any r and $B_r(x_0)$, consider $P_t^r \mathbb{1}(x) := \sum_{y \in B_r} p_t^r(x,y)\,dy$.

Then $P_t^r \mathbb{1}(x) - w(x,t)$ satisfies the heat eq. on

$$\mathrm{Int} B_r \times [0,T).$$

and $P_t^r \mathbb{1} = w(\cdot, 0)$ on $B_r \times \{0\}$

$$P_t^r \mathbb{1} - \underbrace{w(x,t)}_{\text{positive}} < 0 \text{ on } \mathrm{bdy}\, B_r \times [0,T].$$

By maximum principle, we have $P_t^r \mathbb{1} - w(\cdot,t) \leq 0$.

Letting $r \to \infty$, we have

$$P_{t_0} \mathbb{1}(x_0) \leq w(x_0,t_0) < 1.$$

□

(2) ⇒ (3). We first construct a nonzero, bounded solution to

$$(A)\quad \begin{cases} \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), & \forall\ (x,t) \in V \times (0,T) \\ u(x,0) = 0, & \forall\ x \in V \end{cases}$$

for a ~~finite~~ time interval $(0,T)$.

Then, by the proof of (3) ⇒ (1), existence of such a fct is enough to show (1) holds, and hence (1a), that is

$$P_t \mathbb{1} < \mathbb{1}, \quad \forall\ t \in (0,\infty)$$

This provides the function $\mathbb{1} - P_t \mathbb{1}$ which is a nonzero ~~positive~~, bounded solution to

$$(B)\quad \begin{cases} \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), & \forall\ (x,t) \in V \times (0,\infty) \\ u(x,0) = 0, & \forall\ x \in V. \end{cases}$$

for an ~~infinite~~ time interval.

So it remains to construct a solution to (A).

Let f be a positive, bounded λ-harmonic for $\lambda < 0$, whose existence is guaranteed by (2).

Define $w(x,t) := e^{-\lambda t} f(x)$. Then w is positive. and bounded on $V \times [0,T]$ for any finite number T, and satisfies

$$(C)\quad \begin{cases} \frac{\partial w}{\partial t}(x,t) = -\lambda e^{-\lambda t} f(x) = \overset{e^{-\lambda t}\cdot(-\lambda f(x))}{\cancel{\lambda w(x,t)}} = e^{-\lambda t} \Delta f(x) \\ \qquad = \Delta w(x,t) \quad \text{on } (x,t) \in V \times [0,T] \\ w(x,0) = f(x). \quad \forall\ x \in V. \end{cases}$$

Another solution of (c) is given by $P_t f(x)$.

Since $\sup_{x\in V} P_t v(x) \le \sup_{x\in V} v(x)$

and $w(x,t) = e^{-\lambda t} v(x) > v(x)$ for all $t>0$, all x,

we know the two solutions, $w(x,t)$ & $P_t f(x)$ are indeed distinct.

Taking the difference $w(x,t) - P_t f(x)$ yields a nonzero, bounded solution to the equation (A). $\square$

To finish the proof of Thm III.3.2., we ~~are~~ still need to show (2a) $\Rightarrow$ (2).

<u>Lemma III.3.3</u>: Let G be a locally finite infinite graph, D be a finite, conn. subgraph. For any $\lambda<0$. There exists a ~~positive~~ ~~non-negative~~, ~~bounded function f~~ function f on D s.t.

$$\begin{cases} \Delta_D f + \lambda f = 0 \quad \cancel{\text{for any } \lambda<0}, \text{ on int} D. \\ f|_{\text{bdy} D} = 1, \quad \text{and} \quad 0<f\le 1 \text{ on } D. \end{cases}$$

<u>Proof</u>: We prove the existence by ~~direct~~ explicit constructions.

Recall Δ_D is the Dirichlet Laplacian acting on the space $C(D, \text{bdy}D)$ of functions ~~which~~ on D which vanish on the vertex bdy D.

Let us first solve the following equation for $\lambda<0$:

$$(E) \begin{cases} \Delta_D f + \lambda f = 0 \quad \text{on int} D \\ f|_{\text{bdy} D} = 1. \end{cases}$$

Notice that $f \notin C(D, \text{bdy} D)$.

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<u>Define</u>: $\mathbb{1}_D(x) = \begin{cases} 1, & \text{if } x \in D \\ 0, & \text{otherwise}. \end{cases}$ and consider

$$w(x) := f(x) - \mathbb{1}_D(x).$$

Then $w|_{\mathrm{bdy} D} = 0$, i.e., $w \in C(D, \mathrm{bdy} D)$.

Furthermore, we check on $\mathrm{int}\, D$

$$\Delta_D w(x) = \Delta_D f(x) = -\lambda f(x) = -\lambda (w(x) + \mathbb{1}_D(x))$$

That is,

$$(\Delta_D + \lambda I) w(x) = -\lambda \mathbb{1}_{\mathrm{int} D}.$$

Observe that $\lambda < 0$ implies $\Delta_D + \lambda I$ is invertible on $C(D, \mathrm{bdy}\, D)$. (Since all eigenvalues of Δ_D are positive).

So,

$$w = (\Delta_D + \lambda I)^{-1} (-\lambda \mathbb{1}_{\mathrm{int} D})$$

and

$$f = w + \mathbb{1}_D = (\Delta_D + \lambda I)^{-1} (-\lambda \mathbb{1}_{\mathrm{int} D}) + \mathbb{1}_D.$$

This is a solution of (E).

We now claim $0 < f \leq 1$ on D.

- $f > 0$ on D. This is trivial on $\mathrm{bdy}\, D$.

Suppose that $\exists\, x_0 \in \mathrm{int}\, D$ s.t. $f(x_0) \leq 0$, then we may assume x_0 is a minimum for f and

$$\Delta f(x_0) = \frac{1}{d_{x_0}} \sum_{y \sim x_0} (f(y) - f(x_0)) \geq 0.$$

However, $\Delta f(x_0) = \underbrace{-\lambda}_{<0} \underbrace{f(x_0)}_{\leq 0} \leq 0$.

Hence $\Delta f(x_0) = 0$. This implies that $f(y) = f(x_0)$ for all neighbors y of x_0. Repeating this argument, we get $\underset{\leq 0}{f(x) \equiv f(x_0)}$, $\forall x \in D$. This contradicts to the fact that $f|_{\mathrm{bdy} D} = 1 > 0$.

Therefore, we prove $f > 0$ on D.

② $f \leq 1$ on D.

Since $\Delta_D f = \underbrace{-\lambda}_{>0} \underbrace{f}_{>0} > 0$, we have in particular.

$$\Delta f(x) > 0 \quad \text{for any } x \in \mathrm{int}\, D.$$

This tells

$$\max_D f = \max_{\mathrm{bdy} D} f = 1.$$

(If f attains an interior maximum at $x_0 \in \mathrm{int}\, D$, we have $\Delta f(x_0) \leq 0$). □

Now, we're ready for the proof of (2a) ⇒ (2).

(2a) ⇒ (2). We call a function f satisfying

$$\Delta f + \lambda f \geq 0$$

an λ-subharmonic function.

We need show the existence of positive, bdd λ-subharmonic function for any $\lambda < 0$ implies the existence of a positive, bdd, λ-harmonic function. for any $\lambda < 0$.

Let $\{D_i\}_{i=1}^{\infty}$ be any exhaustion of G. (Recall, it means $D_i \subset D_{i+1}$, $\bigcup_{i=1}^{\infty} D_i = G$ and D_i finite connected).

Let f_i be functions constructed in Lemma III.3.3 for D_i. That is
$$\begin{cases} \Delta_{D_i} f_i + \lambda f_i = 0 & \text{on int } D_i \\ f_i|_{\mathrm{bdy} D_i} = 1 \\ 0 < f_i \leq 1 & \text{on } D_i \end{cases}$$

Extend f_i to be 1 outside of D_i. Then we obtain a ~~monotonic~~ sequence of function on V:

$$\{f_i\}_{i=1}^{\infty}.$$

We claim that $f_i \geq f_{i+1}$.

~~Outside of D_{i+1}: trivial.~~

① at bdy D_i: $f_i = 1$ while $f_{i+1} \leq 1$. ✓

② outside of D_i, $f_i = 1$ while $f_{i+1} \leq 1$ ✓

③ at int D_i: Since $f_i - f_{i+1} \Big|_{\text{bdy } D_i} \geq 0$.

Moreover $\Delta(f_i - f_{i+1}) = -\lambda(f_i - f_{i+1})$ on int D_i

Suppose $\exists\, x_0 \in \text{int } D_i$, s.t. $f_i(x_0) < f_{i+1}(x_0)$.

we can assume x_0 is a minimum of $f_i - f_{i+1}$.

Therefore $\Delta(f_i - f_{i+1})(x_0) \geqslant 0$.

However, $\Delta(f_i - f_{i+1})(x_0) = -\lambda \underbrace{(f_i - f_{i+1})(x_0)}_{<0} < 0$ contradiction.

Therefore, we always have $f_i \geq f_{i+1}$.

That is, ~~we have a~~ $\{f_i\}$ is a ~~bou~~ sequence of non-increasing, bounded sequence. Therefore, they converge pointwisely:

Denote $f(x) := \lim_{i \to \infty} f_i(x)$, $\forall x \in V$.

We again have $0 \leq f \leq 1$. and $\Delta f + \lambda f = 0$. on V.

It remains to show that f is <u>positive</u>.

By (2a), we have a positive, bounded ω s.t. $\Delta\omega + \lambda\omega \geq 0$ on G.

We may assume $\omega \leq 1$.

Let us compare f_i and ω on D_i. for given i.

On $\mathrm{int}\, D_i$, we have

$$\Delta(f_i-\omega) = \Delta_D f_i - \Delta\omega = -\lambda f_i - \Delta\omega$$
$$\leq -\lambda(f_i-\omega)$$

We claim $f_i \geq \omega$ on D_i

① on $\mathrm{bdy}\, D_i$, $f_i = 1 \geq \omega$.

② on $\mathrm{int}\, D_i$: Suppose $\exists\, x_0 \in \mathrm{int}\, D_i$ s.t.

$$(f_i-\omega)(x_0) < 0.$$

We can assume x_0 is a minimum of $f_i-\omega$. Then

$$\Delta(f_i-\omega)(x_0) \geq 0$$

However, $\Delta(f_i-\omega)(x_0) \leq \underbrace{-\lambda}_{>0}\underbrace{(f_i-\omega)(x_0)}_{<0} < 0.$

Contradiction!

Therefore, we have $f_i \geq \omega$ on D_i, $\forall i$.

Letting $i \to \infty$, we arrive at

$$f \geq \omega \text{ on } V.$$

Hence f is positive. □

Theorem III 3.4 Any positive function f satisfying

$$\Delta f + \lambda f = 0.$$

for $\lambda < 0$ is unbounded.

Therefore, G is stochastically complete, i.e. $P_t \mathbb{1} = \mathbb{1}, \forall t$.

Proof: Fix a vertex $x_0 \in G$. At x_0, we have

$$\Delta f(x_0) = \frac{1}{d_{x_0}} \sum_{y:y\sim x_0} (f(y)-f(x_0)) = \frac{1}{d_{x_0}} \sum_{y:y\sim x_0} f(y) - f(x_0)$$

$= -\lambda f(x_0)$.

That is, $\frac{1}{d_{x_0}} \sum_{y: y \sim x_0} f(y) = (1-\lambda) f(x_0)$.

Then, there must be a neighbor of x_0, say, x_1 s.t.

$$f(x_1) \geq \frac{1}{d_{x_0}} \sum_{y: y \sim x_0} f(y) = (1-\lambda) f(x_0).$$

Repeating this argument, we get a sequence of distinct vertices

$$x_0 \sim x_1 \sim x_2 \sim \cdots$$

s.t. $f(x_i) \geq (1-\lambda) f(x_{i-1}) \geq \cdots \geq (1-\lambda)^i f(x_0)$.

Recalling $\lambda < 0$, we have

$$\lim_{i \to \infty} f(x_i) = \infty. \qquad \Box$$

<u>Corollary III.3.5</u>: For any bounded function f_0, the bounded solution to

$$\text{⚠} \begin{cases} \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), & \forall (x,t) \in V \times (0, \infty) \\ u(x,0) = f_0(x), & \forall x \in V \end{cases}$$

is unique.