$$= -\lambda f(x_0).$$

That is, $\frac{1}{d_{x_0}} \sum_{y: y\sim x_0} f(y) = (1-\lambda) f(x_0)$.

Then, there must be a neighbor of x_0, say, x_1 s.t.

$$f(x_1) \geq \frac{1}{d_{x_0}} \sum_{y: y\sim x_0} f(y) = (1-\lambda) f(x_0).$$

Repeating this argument, we get a sequence of distinct vertices

$$x_0 \sim x_1 \sim x_2 \sim \cdots$$

s.t. $f(x_i) \geq (1-\lambda) f(x_{i-1}) \geq \cdots \geq (1-\lambda)^i f(x_0)$.

Recalling $\lambda < 0$, we have

$$\lim_{i\to\infty} f(x_i) = \infty. \qquad \square$$

<u>Corollary III.3.5</u>: For any bounded function f_0, the bounded solution to

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), & \forall (x,t) \in V \times (0,\infty) \\ u(x,0) = f_0(x), & \forall x \in V \end{cases}$$

is unique.

(III.4) <u>Curvature and Gradient estimates</u>.

Recall the Bakry-Émery curvature-dimension inequality is well defined on a locally finite infinite graph. (Indeed, the curvature at a vertex x is a local concept).

Recall that we say $CD(K, n)$ holds at $x \in V$ if

$$\Gamma_2(f)(x) \geq \frac{1}{n}(\Delta f(x))^2 + K\,\Gamma(f)(x) \qquad (*)$$

holds for all $f : V \to \mathbb{R}$. Observe that only finitly many values of f is involved in $(*)$.

<u>Theorem</u> III.4: Let $G = (V, E)$ be a locally finite graph. Then

$$CD\left(2, \ \frac{1}{2}t(x) - 1\right), \text{ where } t(x) := \min_{y: y \sim x}\left(\frac{4}{d_y} + \frac{\#(x,y)}{D(x)}\right)$$

$\#(x,y) = \#\{z \in V : z \sim x, z \sim y\}$ for $x \sim y$ and $D(x) = \max_{y: y\sim x} d_y$ is the ~~local~~ maximal vertex degree in $S_1(x)$.

<u>Proof</u>: ~~Recall~~ Let $f \in C_0(V)$ be a function with finite support. Recall that

$$\Gamma(f)(x) = \Gamma(f, f)(x) = \frac{1}{2}\frac{1}{d_x}\sum_{y: y\sim x}(f(y) - f(x))^2.$$

Moreover,

$$\Gamma_2(f)(x) = \Gamma_2(f, f)(x) = \frac{1}{2}\left(\Delta\Gamma(f)(x) - 2\,\Gamma(f, \Delta f)(x)\right).$$

We have

$$\Delta\Gamma(f)(x) = \frac{1}{d_x}\sum_{y: y\sim x}\left(\Gamma(f)(y) - \Gamma(f)(x)\right)$$

$$= \frac{1}{d_x}\sum_{y: y\sim x}\left[\frac{1}{2d_y}\sum_{z: z\sim y}(f(z) - f(y))^2 - \frac{1}{2}\frac{1}{d_x}\sum_{z: z\sim x}(f(z) - f(x))^2\right]$$

$$= \frac{1}{2}\frac{1}{d_x}\sum_{y: y\sim x}\frac{1}{d_y}\sum_{z: z\sim y}(f(z) - f(y))^2 - \frac{1}{2}\frac{1}{d_x}\sum_{z: z\sim x}(f(z) - f(x))^2$$

$$= \frac{1}{2}\frac{1}{d_x}\sum_{y: y\sim x}\frac{1}{d_y}\sum_{z: z\sim y}\Big[(f(z) - f(y) + f(x) - f(y))^2 - 2(f(z) - f(y))(f(x) - f(y)) - (f(x) - f(y))^2\Big] - \frac{1}{2}\frac{1}{d_x}\sum_{z: z\sim x}(f(z) - f(x))^2$$

$$= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(x)-2f(y)+f(z)\right)^2$$

$$-\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(z)-f(y)\right)\left(f(x)-f(y)\right)-\frac{1}{d_x}\sum_{y:y\sim x}\left(f(x)-f(y)\right)^2.$$

$$= \frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(x)-2f(y)+f(z)\right)^2$$

$$-\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(x)-f(y)\right)\left(f(z)-2f(y)+f(x)\right)$$

and

$$2\Gamma(f,\Delta f)(x)=\frac{1}{d_x}\sum_{y:y\sim x}\left(f(y)-f(x)\right)\left(\Delta f(y)-\Delta f(x)\right)$$

$$=-\Delta f(x)^2+\frac{1}{d_x}\sum_{y:y\sim x}\left(f(y)-f(x)\right)\cdot\frac{1}{d_y}\sum_{z:z\sim y}\left(f(z)-f(y)\right)$$

$$=-\left(\Delta f(x)\right)^2-\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(x)-f(y)\right)\left(f(z)-f(y)\right).$$

Combining the above calculation yields

$$2\Gamma_2(f)(x)=\Delta\Gamma(f)(x)-2\Gamma(f,\Delta f)(x)$$

$$=\frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(x)-2f(y)+f(z)\right)^2+\left(\Delta f(x)\right)^2$$

$$-\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(x)-f(y)\right)\left(f(x)-f(y)\right)$$

That is

$$\Gamma_2(f)(x)=Hf(x)-\Gamma(f)(x)+\frac{1}{2}\left(\Delta f(x)\right)^2$$

where $$Hf(x):=\frac{1}{2}\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(x)-2f(y)+f(z)\right)^2$$

can be considered as an Hessian term.

That is, we show $CD(2,-1)$ always holds at x.

Of course, we can dig more from $Hf(x)$:

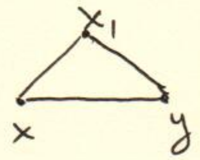
$$Hf(x) = \frac{1}{4}\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\sum_{z:z\sim y}\left(f(z)-2f(y)+f(x)\right)^2$$

$$= \frac{1}{4}\frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\left(4\left(f(x)-f(y)\right)^2 + \sum_{\substack{z:z\sim y\\ z\neq x}}\left(f(z)-2f(y)+f(x)\right)^2\right)$$

$$\geqslant \frac{1}{d_x}\sum_{y:y\sim x}\frac{1}{d_y}\left(f(x)-f(y)\right)^2 \geqslant \frac{2}{D(x)}\Gamma(f)(x).$$

That is, we show $CD\left(2, \frac{2}{D(x)}-1\right)$ always holds at x.

For a given ~~p~~ edge $\{x,y\}$, if there exists x_1 s.t. $x_1\sim x$, $x_1\sim y$.



we have

$$\frac{1}{d_y}\left(f(x_1)-2f(y)+f(x)\right)^2 + \frac{1}{d_{x_1}}\left(f(y)-2f(x_1)+f(x)\right)^2$$

$$\geqslant \frac{1}{D(x)}\Big[\left(f(x_1)-f(y)\right)^2 + \underline{\left(f(x)-f(y)\right)^2} + 2\left(f(x)-f(y)\right)\left(f(x_1)-f(y)\right)$$
$$+ \underline{\left(f(x)-f(x_1)\right)^2} + \left(f(y)-f(x_1)\right)^2 + 2\left(f(y)-f(x_1)\right)\left(f(x)-f(x_1)\right)\Big]$$

$$= \frac{1}{D(x)}\Big[\left(f(x)-f(y)\right)^2 + \left(f(x)-f(x_1)\right)^2 + 2\left(f(x_1)-f(y)\right)^2 + 2\left(f(x)-f(y)\right)\cdot\left(f(x_1)-f(y)\right)\Big]$$

$$= \frac{1}{D(x)}\Big[\left(f(x)-f(y)\right)^2 + \left(f(x)-f(x_1)\right)^2 + 4\left(f(x_1)-f(y)\right)^2\Big]$$

$$\geqslant \frac{1}{D(x)}\Big[\left(f(x)-f(y)\right)^2 + \left(f(x)-f(x_1)\right)^2\Big]$$

Therefore,

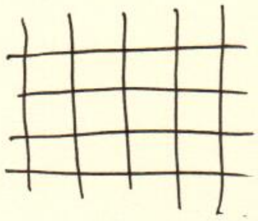
$$Hf(x) \geqslant \frac{1}{4d_x}\sum_{y:y\sim x}\frac{1}{d_y}\left(4\left(f(x)-f(y)\right)^2 + \frac{\#(x,y)}{D(x)}\left(f(x)-f(y)\right)^2\right)$$

$$\geqslant \frac{t(x)}{2}\cdot\Gamma(f)(x).$$ ~~That~~

That is, $CD\left(2, \frac{1}{2}t(x)-1\right)$ always holds at $x \in V$. $\square$

<u>Example</u>: Ricci flat graphs à la Chung and Yau. (1996)

<u>Intuition</u>: lattice $\mathbb{Z}^2$. We can define maps.



$$\eta_1 = \text{Up}: V \longrightarrow V$$
$$\eta_2 = \text{Down}: V \longrightarrow V$$
$$\eta_3 = \text{Left}: V \longrightarrow V$$
$$\eta_4 = \text{Right}: V \longrightarrow V.$$

Then, we have

(i) $\eta_i x \sim x$, $\forall x \in V$, $\forall i$

(ii) $\eta_i x \neq \eta_j x$, $\forall x \in V$, $\forall i \neq j$

(iii) $\eta_j \eta_i x = \eta_i \eta_j x$, $\forall x \in V$, $\forall i, j$.

For abelian Cayley graphs, such maps can be given by its generators. We consider a general flatness notion.

<u>Definition</u> III.4.2. Let $G = (V, E)$ be a d-regular graph. We say $x \in V$ is Ricci flat if there exists maps

$$\eta_i : B_1(x) \to V, \quad i = 1, 2, \dots, d.$$

with the following property:

(i) $\eta_i u \sim u$, $\forall u \in B_1(x)$

(ii) $\eta_i u \neq \eta_j u$, $\forall u \in B_1(x)$, $\forall i \neq j$

(iii) $\bigcup_j \eta_i \eta_j x = \bigcup_j \eta_j \eta_i x$, $\forall i$. $\square$

<u>Rmk</u>. (iii) is an "average" commutability.

For any $f: B_2(x) \to \mathbb{R}$, we define

$$\nabla_j f : B_1(x) \to \mathbb{R}$$

s.t. $\nabla_j f(y) = f(\eta_j y) - f(y), \ \forall y \in B_1(x).$

Then $\forall f : V \to \mathbb{R}$, we have

$$\Delta f(x) = \frac{1}{d}\sum_{i=1}^{d}\left(f(\eta_i x) - f(x)\right) = \frac{1}{d}\sum_{i=1}^{d} \nabla_i f(x).$$

Lemma III.4.3 : ~~For any $f : B_1(x) \to \mathbb{R}$,~~

Property (iii) holds $\Leftrightarrow [\Delta, \nabla_j] f(x) = 0, \ \forall j, \forall f : B_2(x) \to \mathbb{R}.$

Proof: We calculate

$$[\Delta, \nabla_j] f(x) = \Delta(\nabla_j f)(x) - \nabla_j(\Delta f)(x)$$

$$= \frac{1}{d}\sum_{i=1}^{d}\left(\nabla_j f(\eta_i x) - \nabla_j f(x)\right) - \left(\Delta f(\eta_j x) - \Delta f(x)\right)$$

$$= \frac{1}{d}\sum_{i=1}^{d}\left(f(\eta_j \eta_i x) - f(\eta_i x) - f(\eta_j x) + f(x)\right)$$

$$- \frac{1}{d}\sum_{i=1}^{d}\left(f(\eta_i \eta_j x) - f(\eta_j x) - f(\eta_i x) + f(x)\right)$$

$$= \frac{1}{d}\sum_{i=1}^{d}\left(f(\eta_j \eta_i x) - f(\eta_i \eta_j x)\right). \qquad \square$$

Theorem III.4.4: If G is Ricci flat at $x \in V$. Then $CD(0, \infty)$ holds at x.

Proof: (follows Schmuckenschläger 1998). We calculate

$$\Gamma_2(f,f)(x) = \frac{1}{2}\Delta\Gamma(f,f)(x) - \Gamma(f, \Delta f)(x).$$

$$= \frac{1}{2}\Delta\left(\frac{1}{2d}\sum_{i=1}^{d}(\nabla_i f)^2\right)(x) - \frac{1}{2}\frac{1}{d}\sum_{j=1}^{d}\nabla_j f(x)\,\nabla_j(\Delta f)(x).$$

$$= \frac{1}{4d} \sum_{i=1}^{d} \Delta (\nabla_j f)^2 (x) - \frac{1}{2d} \sum_{j=1}^{d} \nabla_j f(x)\, \nabla_j (\Delta f)(x)$$

$$= \frac{1}{2d} \sum_{i=1}^{d} \Big(\Gamma(\nabla_j f, \nabla_j f)(x) + \nabla_j f(x) \cdot \Delta(\nabla_j f)(x) \Big) - \frac{1}{2d} \sum_{j=1}^{d} \nabla_j f(x)\, \nabla_j (\Delta f)(x)$$

$$= \frac{1}{2d} \sum_{i=1}^{d} \Gamma(\nabla_j f, \nabla_j f)(x) + \frac{1}{2d} \sum_{i=1}^{d} \nabla_j f(x)\, \underbrace{[\Delta, \nabla_j] f(x)}_{= 0 \text{ for Ricci flat}}.$$

$$= \frac{1}{2d} \sum_{i=1}^{d} \Gamma(\nabla_j f, \nabla_j f)(x) \geq 0. \qquad \square.$$

<u>Rmk</u>: In fact, if x is Ricci flat, it holds that ~~CD(d, 0)~~ $CD(0, d)$, which implies $CD(0, \infty)$.

We leave this to the readers.

<u>Gradient estimate</u>:

<u>Theorem III.4.5</u>. Let $G = (V, E)$ be a locally finite graph. Then the following are equivalent

(a) $CD(K, \infty)$ holds at each $x \in V$.

(b) $\Gamma(P_t f) \leq e^{-2Kt} P_t(\Gamma(f))$ for any bounded function $f : V \to \mathbb{R}$.

<u>Proof</u>: (a) $\Rightarrow$ (b). First consider $f \in C_0(V)$.

Consider the function

$$F(s) = e^{-2Ks} \underbrace{P_s \Gamma(P_{t-s} f)}_{:= G(s)}, \quad 0 \leq s \leq t.$$

Recall that if f is bounded, so is $\Gamma(f)$.

if $f \in \ell^2(V)$, so is $\Gamma(f) \in \ell^1(V)$.

$F(0) = \Gamma(P_t f)$, $F(t) = e^{-2Kt} P_t \Gamma(f)$.

Next, we hope to calculate $\frac{\partial}{\partial s}F(s)$. For that purpose, we need $\frac{\partial}{\partial s}G(s)$.

Formally, we have

$$\frac{\partial}{\partial s}G(s)(x) = \frac{\partial}{\partial s}\sum_{y\in V} p_s(x,y)\Gamma(P_{t-s}f, P_{t-s}f)(y)\,dy.$$

$$= \sum_{y\in V}\Delta p_s(x,y)\Gamma(P_{t-s}f, P_{t-s}f)(y)\,dy$$

$$+ \sum_{y\in V} p_s(x,y)\underbrace{2\Gamma}_{\text{finite sum}}(P_{t-s}f, -\Delta P_{t-s}f)(y)\,dy.$$

To ensure the above formal derivative is indeed the derivative of $G(s)$, we ~~need~~ can show that the absolute values of the summands are ~~un~~ uniformly (in s) controlled by a summable function.

For any $s\in(0,t)$, we ~~c~~ may assume that $s\in[\varepsilon, t-\varepsilon]$ $\varepsilon>0$.

Then

$$p_s(x,y)\Gamma(P_{t-s}f, P_{t-s}f)(y) \leq \sup_{s\in[\varepsilon,t-\varepsilon]}\Gamma(P_{t-s}f)(y) = h(y).$$

$$p_s(x,y)\left|\Gamma(P_{t-s}f, \Delta P_{t-s}f)(y)\right| \leq \sup_{s\in[\varepsilon,t-\varepsilon]}\Gamma(P_{t-s}f, \Delta P_{t-s}f)(y)$$

$$= g(y).$$

We claim $\sum_{y\in V} h(y)\,dy < \infty$ and $\sum_{y\in V} g(y)\,dy < \infty$.

Notice that $\Delta P_{t-s}f = P_{t-s}\Delta f$ and

$$\cancel{\Gamma(f,g)}\quad \Gamma(P_{t-s}f, P_{t-s}\Delta f)(y) \leq \sqrt{\Gamma(P_{t-s}f)(y)\,\Gamma(P_{t-s}\Delta f)(y)}$$

$$\leq \frac{1}{2}\Big(\Gamma(P_{t-s}f)(y) + \Gamma(P_{t-s}\Delta f)(y)\Big).$$

$$\Rightarrow \sum_{y\in V} g(y)\,dy \leq \frac{1}{2}\Big(\sum_{y\in V}\Gamma(P_{t-s}f)(y)\,dy + \sum_{y\in V}\Gamma(P_{t-s}\Delta f)(y)\,dy\Big)$$