$$= \frac{1}{4d}\sum_{i=1}^{d} \Delta (\nabla_j f)^2(x) - \frac{1}{2d}\sum_{j=1}^{d} \nabla_j f(x)\, \nabla_j(\Delta f)(x)$$

$$= \frac{1}{2d}\sum_{i=1}^{d}\Big(\Gamma(\nabla_j f, \nabla_j f)(x) + \nabla_j f(x)\cdot \Delta(\nabla_j f)(x)\Big) - \frac{1}{2d}\sum_{j=1}^{d} \nabla_j f(x)\, \nabla_j(\Delta f)(x)$$

$$= \frac{1}{2d}\sum_{i=1}^{d}\Gamma(\nabla_j f, \nabla_j f)(x) + \frac{1}{2d}\sum_{i=1}^{d} \nabla_j f(x) \underbrace{[\Delta, \nabla_j] f(x)}_{=0 \text{ for Ricci flat}}.$$

$$= \frac{1}{2d}\sum_{i=1}^{d}\Gamma(\nabla_j f, \nabla_j f)(x) \geq 0. \qquad \square.$$

Rmk: In fact, if x is Ricci flat, it holds that ~~$CD(d,0)$~~ $CD(0,d)$, which implies $CD(0,\infty)$. We leave this to the readers.

Gradient estimate:

Theorem III.4.5. Let $G=(V,E)$ be a locally finite graph. Then the following are equivalent

(a) $CD(K,\infty)$ holds at each $x \in V$.

(b) $\Gamma(P_t f) \leq e^{-2Kt} P_t(\Gamma(f))$ for any bounded function $f: V \to \mathbb{R}$.

Proof: (a) $\Rightarrow$ (b). First consider $f \in C_0(V)$.

Consider the function

$$F(s) = e^{-2Ks} \underbrace{P_s \Gamma(P_{t-s} f)}_{:=G(s)}, \quad 0 \leq s \leq t.$$

Recall that if f is bounded, so is $\Gamma(f)$.

if $f \in \ell^2(V)$, so is $\Gamma(f) \in \ell^1(V)$.

$F(0) = \Gamma(P_t f)$, $F(t) = e^{-2Kt} P_t \Gamma(f)$.

Next, we hope to calculate $\frac{\partial}{\partial s} F(s)$. For that purpose, we need $\frac{\partial}{\partial s} G(s)$.

Formally, we have

$$\frac{\partial}{\partial s} G(s)(x) = \frac{\partial}{\partial s} \sum_{y\in V} p_s(x,y)\Gamma(P_{t-s}f, P_{t-s}f)(y)\,dy.$$

$$= \sum_{y\in V} \Delta p_s(x,y)\Gamma(P_{t-s}f, P_{t-s}f)(y)\,dy$$

$$+ \sum_{y\in V} p_s(x,y) 2 \underset{\substack{\uparrow \\ \text{finite sum}}}{\Gamma} (P_{t-s}f, -\Delta P_{t-s}f)(y)\,dy. \qquad (***)$$

To ensure the above formal derivative is indeed the derivative of $G(s)$, we ~~need~~ can show that the absolute values of the summands are uniformly (in s) controlled by a summable function.

For any $s\in(0,t)$, we may assume that $s\in[\varepsilon, t-\varepsilon]$ $\varepsilon>0$.

Then

$$p_s(x,y)\Gamma(P_{t-s}f, P_{t-s}f)(y) \leq \sup_{s\in[\varepsilon,t-\varepsilon]} \Gamma(P_{t-s}f)(y) = h(y).$$

$$p_s(x,y)\left|\Gamma(P_{t-s}f, \Delta P_{t-s}f)(y)\right| \leq \sup_{s\in[\varepsilon,t-\varepsilon]} \Gamma(P_{t-s}f, \Delta P_{t-s}f)(y)$$

$$= g(y).$$

We claim $\sum_{y\in V} h(y)\,dy < \infty$ and $\sum_{y\in V} g(y)\,dy < \infty$.

Notice that $\Delta P_{t-s}f = P_{t-s}\Delta f$ and

$$\Gamma(P_{t-s}f, P_{t-s}\Delta f)(y) \leq \sqrt{\Gamma(P_{t-s}f)(y)\Gamma(P_{t-s}\Delta f)(y)}$$

$$\leq \frac{1}{2}\left(\Gamma(P_{t-s}f)(y) + \Gamma(P_{t-s}\Delta f)(y)\right).$$

$$\Rightarrow \sum_{y\in V} g(y)\,dy \leq \frac{1}{2}\left(\sum_{y\in V}\Gamma(P_{t-s}f)(y)\,dy + \sum_{y\in V}\Gamma(P_{t-s}\Delta f)(y)\,dy\right)$$

Since $f \in C_0(V) \Rightarrow \Delta f \in C_0(V)$. We need.

Claim: $\sup_{t \in [\varepsilon, b]} \Gamma(P_t f)(x)$ is in $\ell^1(V)$.

Proof:
$$\Gamma(P_t f) = \Gamma(P_\varepsilon f) + \int_\varepsilon^t \frac{d}{ds} \Gamma(P_s f)\, ds$$
$$= \Gamma(P_\varepsilon f) + 2 \int_\varepsilon^t \Gamma(P_s f, \Delta P_s f)\, ds.$$

Therefore

$$\Big\| \sup_{t \in [\varepsilon, b]} \Gamma(P_t f) \Big\|_{\ell^1} = \sum_{x \in V} \sup_{t \in [\varepsilon, b]} \Gamma(P_t f)(x)\, dx$$

$$= \sum_{x \in V} \Gamma(P_\varepsilon f)(x) dx + 2 \sum_{x \in V} \sup_{t \in [\varepsilon, b]} \int_\varepsilon^t \Gamma(P_s f, \Delta P_s f)(x)\, ds\, dx$$

$$\leq \langle P_\varepsilon f, \Delta P_\varepsilon f \rangle + 2 \sum_{x \in V} \int_\varepsilon^b \underbrace{|\Gamma(P_s f, \Delta P_s f)(x)|}_{\leq \sqrt{\Gamma(P_s f) \cdot \Gamma(\Delta P_s f)}\ \leq \frac{1}{2}(\Gamma(P_s f) + \Gamma(\Delta P_s f))} ds\, dx$$

$$\leq \|P_\varepsilon f\|_{\ell^2} \cdot \|\Delta P_\varepsilon f\|_{\ell^2} + 2 \int_\varepsilon^b \sum_{x \in V} |\Gamma(P_s f, \Delta P_s f)(x)|\, dx\, ds$$

$$\leq 2\|f\|_{\ell^2}^2 + \int_\varepsilon^b \Big[\underbrace{\sum_{x \in V} \Gamma(P_s f)(x) dx}_{= \langle P_s f, \Delta P_s f \rangle} + \underbrace{\sum_{x \in V} \Gamma(\Delta P_s f)(x) dx}_{= \langle \Delta P_s f, \Delta^2 P_s f \rangle} \Big] ds$$

$$\leq 2\|f\|_{\ell^2}^2 + \int_\varepsilon^b \Big[2\|f\|_{\ell^2}^2 + 4\|f\|_{\ell^2}^2 \Big] ds$$

$$= 8 \|f\|_{\ell^2}^2 (b - \varepsilon).$$ $\square$.

Therefore, (***) is indeed the derivative of $G(s)$. That is.

$$\frac{\partial}{\partial s} G(s) = \Delta P_s (\Gamma(P_{t-s} f)) - 2 P_s (\Gamma(P_{t-s} f, \Delta P_{t-s} f)).$$

(Notice that $f \in C_0(V) \Rightarrow f$ bdd $\Rightarrow \Gamma(P_{t-s} f)$, $\Gamma(P_{t-s} f, \Delta P_{t-s} f)$ bdd.

Therefore $P_s(\Gamma(P_{t-s} f))$, $P_s(\Gamma(P_{t-s} f, \Delta P_{t-s} f))$ make sense).

In conclusion

$$\frac{\partial}{\partial s} F(s) = -2k\, e^{-2ks} P_s \Gamma(P_{t-s} f) + e^{-2ks} \underbrace{\Delta P_s}_{= P_s \Delta} (\Gamma(P_{t-s} f)) - 2 e^{-2ks} P_s (\Gamma(P_{t-s} f, \Delta P_{t-s} f))$$

$\cancel{= e^{-2ks}}$

Since $\Gamma(P_{t-s} f)$ is bdd and $\Gamma(P_{t-s} f)$ is ℓ^1 $\Big\} \Rightarrow \Gamma(P_{t-s} f)$ is ℓ^2.

$$= e^{-2ks} P_s \Big[-2k\Gamma(P_{t-s} f) + \underbrace{\Delta \Gamma(P_{t-s} f) - 2\Gamma(P_{t-s} f, \Delta P_{t-s} f)}_{2\Gamma_2(P_{t-s} f)} \Big]$$

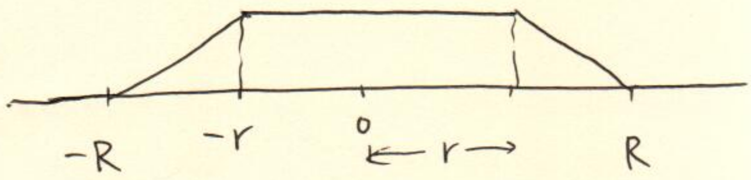
$$= e^{-2ks} P_s \Big[\underbrace{2\Gamma_2(P_{t-s} f) - 2k\Gamma(P_{t-s} f)}_{\geq 0} \Big]$$

$\geq 0.$ by positivity of P_s.

Therefore (b) holds. for any $f \in C_0(V)$ □

Let $o \in V$, $0 < r < R$. Consider the cut-off function

$$\eta_{r,R}(\cdot) = \left(\frac{R - d(\cdot, o)}{R - r} \vee 0 \right) \wedge 1$$



Denote $\eta_k := \eta_{k,2k}$.

Lemma III.4.6. We have $\eta_k \to \mathbb{1}$ pointwisely, $\eta_k \leq \eta_{k+1}$, and $\Gamma(\eta_k) \leq \frac{1}{2k^2}$.

Proof. Observe by triangle inequality that $|\eta_k(y) - \eta_k(x)| \leq \frac{1}{k}$ for $x \sim y$.

Therefore, $$\Gamma(\eta_k)(x) = \frac{1}{2d_x}\sum_{y:y\sim x}\left(\eta_k(y)-\eta_k(x)\right)^2$$
$$\leq \frac{1}{2k^2}. \qquad \square$$

Lemma III.4.7. For any bounded function f, we have
$$P_t \Delta f = \Delta P_t f.$$

Proof. First consider the case $f \in C_0(V)$. We have
$$\Delta(P_t f)(x) = \frac{1}{d_x}\sum_{y:y\sim x}\left(P_t f(y) - P_t f(x)\right)$$
$$= \frac{1}{d_x}\sum_{y:y\sim x}\sum_{z\in V}\left(p_t(y,z) - p_t(x,z)\right)f(z)\,d_z \quad ①$$

On the other hand,
$$P_t(\Delta f)(x) = \sum_{z\in V} p_t(x,z)\,\Delta f(z)$$
Green's thm Lemma III.2.3
$$= \sum_{z\in V} \Delta_z\, p_t(x,z)\, f(z)$$
$$= \sum_{z\in V} \Delta_x\, p_t(x,z)\, f(z)$$
$$= \sum_{z\in V}\frac{1}{d_x}\sum_{y:y\sim x}\left(p_t(y,z) - p_t(x,z)\right)f(z) \quad ②$$

Comparing ① and ②, we derive
$$\Delta(P_t f)(x) = P_t(\Delta f)(x) \quad \text{for } f \in C_0(V).$$

For any bounded function f, we have $\eta_k f \in C_0(V)$.

Hence $\Delta P_t(\eta_k f) = P_t \Delta(\eta_k f)$
$$\lim_{k\to\infty}\Delta P_t(\eta_k f) = \Delta \lim_{k\to\infty} P_t(\eta_k f) = \Delta\cdot\lim_{k\to\infty}\sum_{y\in V} p_t(\cdot,y) f(y)\eta_k(y)\,d_y$$

Since $|p_t(\cdot,y) f(y)\eta_k(y)| \leq \|f\|_\infty\, p_t(\cdot,y)$ which is in $\ell^1(V)$.

By dominant convergence, we have $\lim_{k\to\infty}\Delta P_t(\eta_k f) = \Delta P_t(\lim_{k\to\infty}\eta_k f) = \Delta P_t(f)$.

Similarly, we have
$$\lim_{k\to\infty} P_t \Delta(\eta_k f) = P_t \Delta f.$$
Therefore, we have
$$\Delta P_t f = P_t \Delta f \quad \text{for any bounded fct } f. \quad \square$$

(a) $\Rightarrow$ (b) for any $f \in \ell^\infty(V)$.

Since $\eta_k f \in C_0(V)$, we have
$$\Gamma(P_t(\eta_k f)) \le e^{-2Kt} P_t(\Gamma(\eta_k f))$$
Notice that
$$P_t(\Gamma(\eta_k f)) = \sum_{y\in V} P_t(\cdot,y)\,\Gamma(\eta_k f)(y)\, d_y$$
where
$$\Gamma(\eta_k f)(y) = \frac{1}{2d_y}\sum_{z\sim y}\left(\eta_k(z)f(z) - \eta_k(y)f(y)\right)^2$$
$$\le 2\|f\|_\infty^2$$
That is, $P_t(\cdot,y)\Gamma(\eta_k f)(y) \le 2\|f\|_\infty^2 \cdot P_t(\cdot,y)$

which is ℓ^1.

By dominant convergence theorem, we have
$$\lim_{k\to\infty} P_t(\Gamma(\eta_k f)) = P_t\Gamma(\lim_{k\to\infty}\eta_k f) = P_t\Gamma(f).$$
Similarly, $\lim_{k\to\infty}\Gamma(P_t(\eta_k f)) = \Gamma(P_t(f))$.

Therefore, $\Gamma(P_t f) \le e^{-2Kt} P_t\Gamma(f)$. for any $f\in\ell^\infty(V)$. $\square$

(b) $\Rightarrow$ (a). For any given $x\in V$, consider
$$F(t) := e^{-2Kt}P_t(\Gamma(f))(x) - \Gamma(P_t f)(x); \quad \forall f\in C_0(V).$$
Note that $F(0) = 0$. By (b), we have $F \ge 0$.

Observe that F is differentiable, ~~and~~ Hence $F'(0) \ge 0$.

We have

$$\frac{d}{dt}\Big|_{t=0} P_t(\Gamma(f))(x) = \Delta P_t(\Gamma(f))(x)\Big|_{t=0} = \Delta(\Gamma(f)(x)).$$

and

$$\frac{d}{dt}\Big|_{t=0} \Gamma(P_t f)(x) = 2\Gamma(P_t f, \Delta P_t f))(x)\Big|_{t=0} = 2\Gamma(f, \Delta f)(x)$$

This yields

$$F'(0) = \frac{d}{dt}\Big|_{t=0} F(t) = -2K\Gamma(f)(x) + \Delta(\Gamma(f))(x) - 2\Gamma(f,\Delta f)(x)$$

$$= 2\Gamma_2(f)(x) - 2K\Gamma(f)(x)$$

That is,

$$F'(0) \geq 0 \text{ implies } \Gamma_2(f)(x) \geq K\Gamma(f)(x). \qquad \square.$$

To conclude ~~ude~~ this section, we give an alternative proof of the stochastic completeness..

<u>Proof</u> <u>of</u> <u>stochastic</u> <u>completeness</u> by <u>curvature</u> :

It suffices to show $P_t \mathbb{1} = \mathbb{1}$.

For any $x \in V$, and any $t > 0$, we have

$$\Gamma(P_t \mathbb{1})(x) = \Gamma\left(P_t \lim_{k\to\infty} \eta_k\right)(x)$$

$$\overset{\text{dominant convergence}}{=} \lim_{k\to\infty} \Gamma(P_t \eta_k)(x)$$

$$\leq \liminf_{k\to\infty} e^{-2Kt} P_t(\underbrace{\Gamma(\eta_k)}_{\leq \frac{1}{2k^2}})(x)$$

$$\leq \liminf_{k\to\infty} \frac{1}{2k^2} e^{-2Kt} = 0$$

where we use ~~any~~ Thm 4.1 (any graph satisfies $CD(\cancel{?}, \cancel{?})$ $\Rightarrow CD(-1,\infty)$. (We can always set $K=-1$))

This means, for any given $t>0$, $P_t\mathbb{1}$ is a constant function on V.

(III.5) Diameter bounds of graphs.

Theorem III.5.1 (L.-Münch-Peyerimhoff 2018)
Let $G=(V,E)$ be a locally finite connected graph satisfying $CD(K,\infty)$, $K>0$. Then we have

$$d(x,y)\leq \frac{2}{K}, \text{ for any } x,y\in V.$$

That is, the diameter $\mathrm{diam}(G)=\sup_{x,y\in V} d(x,y)\leq \frac{2}{K}$.
Particularly, the graph G is finite.

Remark. On Rie. mflds, such estimates need the dimension n. Here, we have ~~the~~ such an estimate on graphs even ~~if~~ $n=\infty$!! The following lemma makes graph setting so special.

Lemma III.5.2 For any $f:V\to\mathbb{R}$ and any $x\in V$, we have
$$\left(\Delta f(x)\right)^2\leq 2\Gamma(f)(x).$$

Proof: $$\left(\Delta f(x)\right)^2=\left(\frac{1}{d_x}\sum_{y\sim x}(f(y)-f(x))\right)^2=\left(\sum_{y\sim x}\frac{f(y)-f(x)}{\sqrt{d_x}}\cdot\frac{1}{\sqrt{d_x}}\right)^2$$
$$\leq \sum_{y\sim x}\left(\frac{f(y)-f(x)}{\sqrt{d_x}}\right)^2\cdot\underbrace{\sum_{y\sim x}\left(\frac{1}{\sqrt{d_x}}\right)^2}_{=1}$$
$$=\frac{1}{d_x}\sum_{y\sim x}(f(y)-f(x))^2=2\Gamma(f)(x).$$ □.

Rmk. This lemma can not hold in the continuous setting. We can explain why diameter estimate still hold for graphs even if $n=\infty$ as below.

Recall $CD(K,n)$ means

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta f)^2 + K\,\Gamma(f), \quad \forall f.$$

By the Lemma III.5.2, we ~~have~~ can derive from

$$CD(K,\infty) : \Gamma_2(f) \geq K\,\Gamma(f)$$

that

$$\Gamma_2(f) \geq \frac{c}{2}(\Delta f)^2 + (K-c)\,\Gamma(f).$$

That is, we can "produce" the dimension term from the curvature term !! This is the underlying philosophy.

<u>Proof of Thm III.5.1</u>. For any given $x, y \in V$, consider the fct $f : V \to \mathbb{R}$ s.t. $f(\cdot) := d(\cdot, y)$.

Then $d(x,y) = f(x) - f(y)$.

However, f is unbounded. We define

$$h(z) = \begin{cases} R, & \text{if } d(z,y) \geq R \\ d(z,y) & \text{if } d(z,y) < R \end{cases}, \quad R > d(x,y).$$

Then h is bounded, and

$$\begin{aligned} d(x,y) &= h(x) - h(y) \\ &= h(x) - P_t h(x) + P_t h(x) - P_t h(y) + P_t h(y) - h(y) \\ &\leq |h(x) - P_t h(x)| + |P_t h(x) - P_t h(y)| + |P_t h(y) - h(y)|. \end{aligned}$$

~~The~~ <u>Claim</u>. $|h(x) - P_t h(x)| \leq \frac{1}{K}, \ \forall t > 0.$

Indeed, we have

$$|h(x) - P_t h(x)| = \left| \int_0^t \frac{\partial}{\partial s} P_s h(x)\, ds \right|$$

$$= \left| \int_0^t \Delta P_s h(x)\, ds \right| \leq \int_0^\infty |\Delta P_s h(x)|\, ds$$

$$\overset{\text{Lemma III 5.2}}{\leq} \int_0^\infty \sqrt{|2\Gamma(P_s h)(x)|}\, ds.$$

$$\overset{CD(K,\infty)}{\leq} \int_0^\infty \sqrt{2 e^{-2Ks} P_s(\Gamma(h))(x)}\, ds.$$

Observe that $\Gamma(h)(z) = \frac{1}{2d_z} \sum_{y: y\sim z} (h(y) - h(z))^2 \leq \frac{1}{2}$

We have $P_s(\Gamma(h))(x) \leq \frac{1}{2}$

Therefore, $|h(x) - P_t h(x)| \leq \int_0^\infty e^{-Ks}\, ds = \frac{1}{K}$. $\square$

Inserting the Claim ~~into~~ back, we derive

$$d(x,y) \leq \frac{2}{K} + |P_t h(x) - P_t h(y)|. \quad (**)$$

Now, we use again $CD(K,\infty)$ which tells

$$\Gamma(P_t h)(x) \leq e^{-2Kt} P_t(\Gamma h)(x) \leq \frac{1}{2} e^{-2Kt}$$

$\to 0$ as $t \to \infty$.

Therefore $|P_t h(x) - P_t h(y)| \to 0$ as $t \to \infty$ by connectivity of G.

Letting $t \to \infty$ in $(**)$ yields $d(x,y) \leq \frac{2}{K}$. $\square$

Remark: In fact, it holds true that: For locally infinite graphs with $CD(K,\infty)$, $K>0$,

$\mathrm{diam}(G) = \frac{2}{K}$ holds iff G is an n-Hypercube for some n.

For Hyperaube H^n, we have $\mathrm{diam} = n$
and $K = \frac{2}{n}$.

(L., Münch, Peyerimhoff arXiv: 1705.06789)

A natural question is:

If G satisfies $CD(K,\infty)$, $K>0$ at vertices in $V \setminus V_0$, where $|V_0| < \infty$, is G finite?