$$|h(x) - P_t h(x)| = \left| \int_0^t \frac{\partial}{\partial s} P_s h(x)\, ds \right|$$

$$= \left| \int_0^t \Delta P_s h(x)\, ds \right| \leq \int_0^\infty |\Delta P_s h(x)|\, ds$$

$$\overset{\text{Lemma III 5.2}}{\leq} \int_0^\infty \sqrt{|2\Gamma(P_s h)(x)|}\, ds.$$

$$\overset{CD(K,\infty)}{\leq} \int_0^\infty \sqrt{2 e^{-2Ks} P_s(\Gamma(h))(x)}\, ds.$$

Observe that $\Gamma(h)(z) = \frac{1}{2d_z} \sum_{y: y\sim z} (h(y) - h(z))^2 \leq \frac{1}{2}$

We have $P_s(\Gamma(h))(x) \leq \frac{1}{2}$

Therefore, $|h(x) - P_t h(x)| \leq \int_0^\infty e^{-Ks}\, ds = \frac{1}{K}$. □

Inserting the Claim ~~into~~ back, we derive

$$d(x,y) \leq \frac{2}{K} + |P_t h(x) - P_t h(y)|. \qquad (**)$$

Now, we use again $CD(K,\infty)$ which tells

$$\Gamma(P_t h)(x) \leq e^{-2Kt} P_t(\Gamma h)(x) \leq \frac{1}{2} e^{-2Kt} \to 0 \text{ as } t \to \infty.$$

Therefore $|P_t h(x) - P_t h(y)| \to 0$ as $t \to \infty$ by connectivity of G.

Letting $t \to \infty$ in $(**)$ yields $d(x,y) \leq \frac{2}{K}$. □

<u>Remark</u>: In fact, it holds true that: For ~~locally infinite~~ regular !! graphs with $CD(K,\infty)$, $K>0$,

$\operatorname{diam}(G) = \frac{2}{K}$ holds iff G is an n-Hypercube for ~~some~~ $n = \frac{2}{K}$.

For Hyperscube H^n, we have $\mathrm{diam} = n$
and $K = \frac{2}{n}$.
(L., Münch, Peyerimhoff arXiv: 1705.06789)

A natural question is:

If G satisfies $CD(K,\infty)$, $K>0$ at vertices in $V \setminus V_0$, where $|V_0| < \infty$, is G finite?

(III.6). Rigidity results.

In this section we aim at establishing the following rigidity result:

Theorem III.6.1. Let $G=(V,E)$ be a locally finite graph, ~~Let~~ $K>0$. ~~$CD(K,\infty)$, $K>0$~~. Then TFAE.

(1) G is regular, $CD(K,\infty)$ and $\mathrm{diam}(G) = \frac{2}{K}$ (2) G is an n-hypercube where $n = \frac{2}{K}$.

(3) G satisfies $CD(K,\infty)$, $\lambda_{d_{max}} = K$.

Lemma III.6.2. Let $G=(V,E)$ satisfy $CD(K,\infty)$, $K>0$. Let $x_0 \in V$ and $f_0 := d(x_0,\cdot)$. If there exists $y_0 \in V$ s.t.

$$d(x_0,y_0) = \frac{2}{K},$$

then
$$\Gamma(P_t f_0)(x_0) = e^{-2Kt} P_t(\Gamma(f_0))(x_0), \quad \forall\, t \geq 0$$

Proof: By our assumption and the diameter estimate from last section, G is finite. Then similar to last section, we have

$$\frac{2}{K} = \mathrm{diam}(G) = f_0(y_0) - f_0(x_0)$$

$$= f_0(y_0) - P_t f_0(y_0) + \underbrace{P_t f_0(y_0) - P_t f_0(x_0)}_{\to 0 \text{ as } t \to \infty} + P_t f_0(x_0) - f_0(x_0), \ \forall t \geq 0$$

For finite graph,

$$P_t f_0 = \sum_{i=1}^{N} e^{-\lambda_i t} \langle f, \phi_i \rangle \phi_i$$

$$\to \langle f, \phi_0 \rangle \phi_0 \text{ is const.}$$

Therefore

$$\frac{2}{K} = \mathrm{diam}(G) = f_0(y_0) - f_0(x_0)$$

$$= f_0(y_0) - \lim_{t\to\infty} P_t f_0(y_0) + \lim_{t\to\infty} P_t f_0(x_0) - f_0(x_0)$$

$$\leq \int_0^\infty |\partial_t P_t f_0(y_0)| + |\partial_t P_t f_0(x_0)| \, dt$$

$$= \int_0^\infty \left(|\Delta P_t f_0(y_0)| + |\Delta P_t f_0(x_0)| \right) dt$$

$$\leq \int_0^\infty \left(\sqrt{2\Gamma(P_t f_0)(y_0)} + \sqrt{2\Gamma(P_t f_0)(x_0)} \right) dt$$

$$\overset{CD(K,\infty)}{\leq} \sqrt{2} \int_0^\infty e^{-Kt} \left(\sqrt{P_t \Gamma(f_0)(x_0)} + \sqrt{P_t \Gamma(f_0)(y_0)} \right) dt$$

$$\leq \sqrt{2} \int_0^\infty e^{-Kt} \, 2\sqrt{\|\Gamma(f_0)\|_\infty} \, dt$$

$$\leq \sqrt{2} \int_0^\infty e^{-Kt} \, 2 \cdot \sqrt{\frac{1}{2}} \, dt$$

$$= 2 \int_0^\infty e^{-Kt} dt = \frac{2}{K}$$

Hence, each inequality above is an equality. In particular, we have $\Gamma(P_t f_0)(x_0) = e^{-2Kt} P_t \Gamma(f_0)(x_0), \ \forall \ t \geq 0$.

Next, we show.

<u>Lemma III.6.3</u>: Let $G=(V,E)$ satisfy $CD(K,\infty)$, with $x_0 \in V$. Let f be a function $f: V \to \mathbb{R}$. If $\Gamma P_t f(x_0) = e^{-2Kt} P_t \Gamma f(x_0)$, $\forall t \geq 0$

then $\Gamma_2(f) = K\Gamma(f)$.

<u>Proof</u>: Set $F(s) = e^{-2Ks} P_s(\Gamma(P_{t-s}f))(x_0)$.

Then we have $F(0) = \Gamma(P_t f)(x_0)$, $F(t) = e^{-2Kt} P_t \Gamma(f)(x_0)$

We compute

$$\frac{d}{ds}F(s) = e^{-2Ks} P_s \Big(\underbrace{2\Gamma_2(P_{t-s}f) - 2K\Gamma(P_{t-s}f)}_{\geq 0 \text{ by } CD(K,\infty)}\Big)(x_0)$$

However $F(0) = F(t)$ by our assumption. This forces

$$0 = F(t) - F(0) = \int_0^t \frac{d}{ds} F(s)\, ds$$

$$= \int_0^t e^{-2Ks} P_s \big(2\Gamma_2(P_{t-s}f) - 2K\Gamma(P_{t-s}f)\big)(x_0)\, ds$$

$$\geq 0$$

$$\overset{\text{forces}}{\Longrightarrow} e^{-2Ks} P_s \big(2\Gamma_2(P_{t-s}f) - 2K\Gamma(P_{t-s}f)\big)(x_0) = 0, \ \forall s \in [0,t]$$

$$\Rightarrow P_t \big(2\Gamma_2(f) - 2K\Gamma(f)\big)(x_0) = 0.$$

$$\| $$

$$\sum_{y \in V} \underbrace{P_t(x_0,y)}_{\substack{>0 \text{ for any} \\ t>0}} \underbrace{\big(2\Gamma_2(f) - 2K\Gamma(f)\big)(y)}_{\geq 0 \text{ by } CD(K,\infty)} d_y$$

This forces

$$2\Gamma_2(f) = 2K\Gamma(f)(y), \ \forall y \in V. \qquad \square$$

In conclusion, we have.

Lemma III.6.4. Let $G=(V,E)$ satisfy $CD(K,\infty)$, $K>0$. Let $x_0 \in V$ and $f_0 := d(x_0,\cdot)$. If $\exists y_0 \in V$ s.t. $f_0(y_0) = \frac{2}{K}$. Then $\Gamma_2(f_0) = K\Gamma(f_0)$.

Lemma III.6.4'. Let $G=(V,E)$ sat. $CD(K,\infty)$, $K>0$. If $\mathrm{diam}(G) = \frac{2}{K}$, then $\exists x_0 \in V$ s.t. $f_0(\cdot) = d(x_0,\cdot)$ ~~and~~ satisfies $\Gamma_2(f_0) = K\Gamma(f_0)$

For those functions which "attains" the curvature lower bound K, we have the following important observation: It is "determined" by its values on $B_1(x_0)$.

Let us write out the local matrix of $\Gamma(x)$ and $\Gamma_2(x)$. Denote

$$S_1(x) = \{y_1, \cdots, y_m\}$$
$$S_2(x) = \{z_1, \cdots, z_n\}$$

Then $\Gamma(f,g)(x) = \vec{f}^{\top}\Gamma(x)\vec{g}$, where $\vec{f} = \begin{pmatrix} f(x) \\ f(y_1) \\ \vdots \\ f(y_m) \end{pmatrix}$.

and $\Gamma_2(f,g)(x) = \vec{f}^{\top}\Gamma_2(x)\vec{g}$, where $\vec{f} = \begin{pmatrix} f(x) \\ f(y_1) \\ \vdots \\ f(y_m) \\ f(z_1) \\ \vdots \\ f(z_n) \end{pmatrix}$.

For example,

$$2\Gamma(f,g)(x) = \frac{1}{d_x}\sum_{y\in S_1(x)} (f(y)-f(x))\,(g(y)-g(x))$$

$$= \frac{1}{d_x}\sum_{y\in S_1(x)} \Big(f(y)g(y) - f(x)g(y) - f(y)g(x) + f(x)g(x)\Big)$$

Which is translated into

$$\Gamma(x) = \frac{1}{2d_x}\begin{pmatrix} d_x & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Similarly, one can figure out $\Gamma_2(x)$, whose calculation is a bit messy. We can denote

$$\Gamma_2(x) = \begin{pmatrix} \Gamma_2(x)_{xx} & \Gamma_2(x)_{x,S_1(x)} & \Gamma_2(x)_{x,S_2(x)} \\ \Gamma_2(x)_{S_1(x),x} & \Gamma_2(x)_{S_1(x),S_1(x)} & \Gamma_2(x)_{S_1(x),S_2(x)} \\ \Gamma_2(x)_{S_2(x),x} & \Gamma_2(x)_{S_2(x),S_1(x)} & \Gamma_2(x)_{S_2(x),S_1(x)} \end{pmatrix}.$$

$\Gamma_2(x)$ is a symmetric matrix. Moreover

$$\Gamma_2(x)_{S_2(x),S_2(x)} = \frac{1}{4d_x}\begin{pmatrix} \sum\limits_{\substack{y\in S_1(x)\\ y\sim z_1}} \frac{1}{d_y} & & 0 \\ & \ddots & \\ 0 & & \sum\limits_{\substack{y\in S_1(x)\\ y\sim z_n}} \frac{1}{d_y} \end{pmatrix}.$$

Note that, if we denote $p_{xy} = \frac{a_{xy}}{d_x}$, where $a_{xy} = 1$ if $x\sim y$, 0, otherwise.

we can rewrite

$$\frac{1}{d_x}\sum_{\substack{y\in S_1(x)\\ y\sim z}} \frac{1}{d_y} = \sum_{y\in S_1(x)} p_{xy}p_{yz}.$$

Another simple block is

$$\Gamma_2(x)_{S_1(x)S_2(x)} = \left(\Gamma_2(x)_{yz}\right)_{y\in S_1(x),\, z\in S_2(x)},$$ where

$$\Gamma_2(x)_{yz} = -\frac{1}{2}p_{xy}p_{yz} = \begin{cases} -\frac{1}{2}\frac{1}{d_x}\frac{1}{d_y}, & \text{if } y\sim z. \\ 0, & \text{otherwise.}\end{cases}$$

Let $f: V\to \mathbb{R}$ be nonzero such that

$$\Gamma_2(f)(x) = K\Gamma(f)(x)$$

Then we have $\dfrac{\vec{f}^T(\Gamma_2(x)-K\Gamma(x))\vec{f}}{\vec{f}^T\vec{f}} = 0 \Rightarrow \vec{f}$ is an eigenvector of

$\Gamma_2(x)-K\Gamma(x)$ ~~sof~~.

$$\begin{pmatrix} \Gamma_2(x)_{B_1(x),B_1(x)} - K\Gamma(x) & \Gamma_2(x)_{B_1(x),S_2(x)} \\ \Gamma_2(x)_{S_2(x),B_1(x)} & \Gamma_2(x)_{S_2(x),S_2(x)} \end{pmatrix} \begin{pmatrix} \vec{f}\,\mathring{}_{B_1(x)} \\ f(z_1) \\ \vdots \\ f(z_n) \end{pmatrix} = 0.$$

Therefore $\Gamma_2(x)_{S_2(x),B_1(x)} \vec{f}_{B_1(x)} + \underbrace{\Gamma_2(x)_{S_2(x),S_2(x)}}_{\text{invertible.}} \begin{pmatrix} f(z_1) \\ \vdots \\ f(z_n) \end{pmatrix} = 0.$

If $\vec{f}_{B_1(x)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, we have

That is, we obtain: $-\frac{1}{2}\sum_{y\in B_1(x)} p_{xy}p_{yz} + \frac{1}{4}\sum_{y\in S_1(x)} p_{xy}p_{yz} \cdot f(z) = 0 \Rightarrow f(z) = 2.$

<u>Lemma III. 6.5</u>. Let $f : V \to \mathbb{R}$ be a nonzero function s.t.

$\Gamma_2(f)_{(x)} = K\Gamma(f)_{(x)}$ Then $f|_{S_2(x)}$ is "determined" by

$f|_{B_1(x)}$. If $f|_{B_1(x)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, then $f|_{S_2(x)} = \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}$

Indeed, we can provide ~~exact~~ explicit expression of $f|_{S_2(x)}$ interms of $f|_{B_1(x)}$. Note that, those expressions do not depend on the number K at all.

Before proceed further for the <u>diameter</u> sharpness, we mention similar phenomena for the <u>eigenvalue sharpness</u>:

<u>Lemma III. 6.6</u>. Let G satisfy $CD(K,\infty)$, $K>0$. ~~Then~~ Let K be ~~the~~ ~~f~~ an eigenvalue of Δ with an eigenfunction φ, i.e. $\Delta\varphi + K\varphi = 0$. Then

(i) $\Gamma_2(\varphi,\varphi)(x) = K\,\Gamma(\varphi,\varphi)(x)$, $\forall x$.

(ii) $\Gamma(\varphi,\varphi) \equiv \text{const}$.

Proof: Recall that

$$\langle \Gamma(f,g), 1\rangle = -\langle \Delta f, g\rangle, \quad \forall\, f,g: V \to \mathbb{R}$$

Then $K^2 \langle \varphi, \varphi\rangle = \langle K\varphi, K\varphi\rangle = \langle \Delta\varphi, \Delta\varphi\rangle$

$$= -\langle \Gamma(\varphi, \Delta\varphi), 1\rangle = \langle \tfrac{1}{2}\Delta\Gamma(\varphi,\varphi) - \Gamma(\varphi,\Delta\varphi), 1\rangle$$

$$= \langle \Gamma_2(\varphi,\varphi), 1\rangle \overset{\circledast}{\geq} K\langle \Gamma(\varphi,\varphi), 1\rangle$$

Observe $\Gamma_2(\varphi) \geq K\Gamma(\varphi)$ $\quad = -K\langle \varphi, \Delta\varphi\rangle = K^2\langle \varphi,\varphi\rangle.$

This forces $\Gamma_2(\varphi,\varphi)(x) = K\Gamma(\varphi,\varphi)(x), \ \forall x \in V.$

This proves (i).

Moreover, $K\Gamma(\varphi,\varphi) \equiv \Gamma_2(\varphi,\varphi) = \frac{1}{2}\Delta\Gamma(\varphi,\varphi) - \Gamma(\varphi,\Delta\varphi)$

$$= \frac{1}{2}\Delta\Gamma(\varphi,\varphi) + K\Gamma(\varphi,\varphi)$$

$\Rightarrow \frac{1}{2}\Delta\Gamma(\varphi,\varphi) = 0 \Rightarrow \Gamma(\varphi,\varphi) \equiv \text{const.} \Rightarrow$ (ii). $\square$

Combining Lemma III 6.5 & 6.6, we conclude the locality

Lemma III 6.7: Let G satisfy $CD(K,\infty)$. $K>0$, let φ_1, φ_2 be eigenfcts to K, and $\varphi_1|_{B_1(x)} = \varphi_2|_{B_1(x)}$

Then $\varphi_1 \equiv \varphi_2$.