Recall that $\lambda_2 \geq K$ by Lichnerowicz estimate.

Therefore each summand $\lambda_i(\lambda_i - K)\langle f_0, \varphi_i\rangle^2$, $i \geq 2$ is nonnegative. (The first summand $\lambda_1(\lambda_1 - K)\langle f_0, \varphi_1\rangle^2 = 0$)

This forces $\lambda_i(\lambda_i - K)\langle f_0, \varphi_i\rangle^2 = 0$, $\forall i = 2, \cdots, N$.

$\Rightarrow \langle f_0, \varphi_i\rangle = 0$, whenever $i = 2, \cdots, N$ and $\lambda_i \neq K$.

Therefore $f_0 = \varphi + C$ with $\Delta\varphi + K\varphi = 0$ and constant C. □

Rmk: Indeed, one can show (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iii') ; where

(iii) : $\Gamma(P_t f_0) = e^{-2Kt} P_t \Gamma(f_0)$ at $x_0 \in V$

(iii') : $\Gamma(P_t f_0) = e^{-2Kt} P_t \Gamma(f_0)$ at any $x \in V$.

We leave it as an exercise. □.

What is special for distance function?

For given $x_0 \in V$, we define $\frac{a_{zy}}{d_z} = \frac{1}{d_z}$ $z \sim y$

$$\forall z \in V, \quad d_-^{x_0}(z) := \sum_{d(y, x_0) < d(z, x_0)} p_{zy}$$

$$\forall z \in V \quad d_+^{x_0}(z) := \sum_{d(y, x_0) > d(z, x_0)} p_{zy}$$

Let us introduce the N-dim hypercube shell structure:

Def. (N-dim Hypercube shell structure HSS (N, x_0)).

We say a graph $G = (V, E)$ has the Hypercube shell structure HSS (N, x_0) with dimension $N \in (0, \infty)$ w.r.t. $x_0 \in V$, if

(1) G is bipartite

(2) $d_-^{x_0}(x) = \frac{d(x_0, x)}{N}$, $\forall x \in V$.

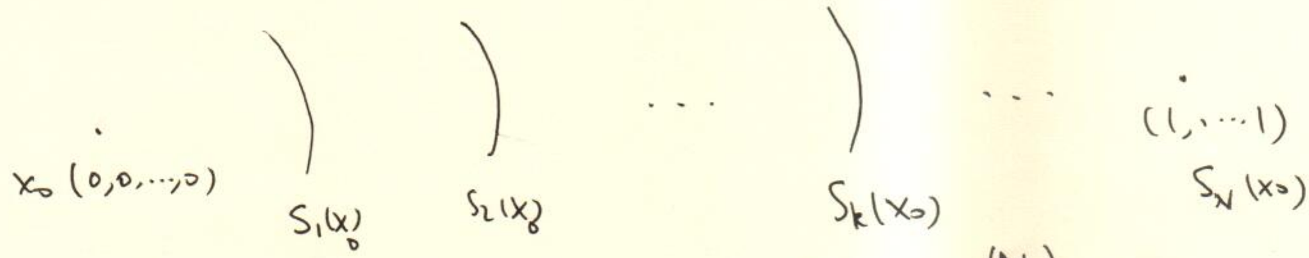
Why the name "hypercube"?

H^N is the graph with vertex set

$$V = \{(x_1, \dots, x_N) : x_i \in \{0,1\}\}$$

$$(x_1, \dots, x_N) \sim (y_1, \dots, y_N) \text{ iff } \sum_{i=1}^{N} |x_i - y_i| = 1.$$

For an N-hypercube, take $x_0 = (0, \dots, 0)$ for example,



$$|S_k(x_0)| = \binom{N}{k}$$

vertex degree $= N = \text{diam}(H^N)$

What can we read from HSS (N, x_0)?

① G is bipartite $\Rightarrow$ $1 = p_-^{x_0}(x) + p_+^{x_0}(x)$.

By (1), we have $p_+^{x_0}(x) = \dfrac{N - d(x_0, x)}{N}$

② For $x \in S_1(x_0)$, (2) tells

$$\frac{1}{d_x} \overset{\text{by def}}{=} p_-^{x_0}(x) \overset{(2)}{=} \frac{1}{N}, \text{ that is, } \underline{d_x = N, \ \forall x \in S_1(x_0).}$$

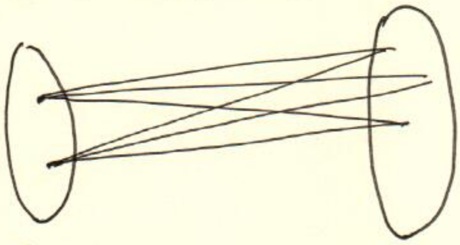
③ We can determine the volume of each "shell" $S_k(x_0)$.

$$\left.\begin{aligned} k=0, &\quad vol(\{x_0\}) = d_{x_0} \\ k=1, &\quad vol(S_1(x_0)) = d_{x_0} \cdot N \end{aligned}\right\} \Rightarrow \frac{vol(S_1(x_0))}{vol(S_0(x_0))} = N.$$

In general, we claim

$$\frac{vol(S_{k+1}(x_0))}{vol(S_k(x_0))} = \frac{N-k}{k+1}. \qquad \text{☆}$$

<u>Proof</u>: We count the number of edges in between $S_k(x_0)$ and $S_{k+1}(x_0)$ in two different ways:



$S_k(x_0)$ $\qquad$ $S_{k+1}(x_0)$

<u>1^{st} way</u>:

$$|E(S_k(x_0), S_{k+1}(x_0))| = \sum_{y \in S_{k+1}(x_0)} p_-^{x_0}(y)\, d_y$$

$$\overset{(2)}{=} \sum_{y \in S_{k+1}(x_0)} \frac{d(x_0,y)}{N}\, d_y$$

$$= \sum_{y \in S_{k+1}(x_0)} \frac{k+1}{N}\, d_y = \frac{k+1}{N} \mathrm{vol}(S_{k+1}(x_0))$$

2^{nd} way:

$$|E(S_k(x_0), S_{k+1}(x_0))| = \sum_{y \in S_k(x_0)} p_+^{x_0}(y)\, d_y$$

$$= \sum_{y \in S_k(x_0)} \frac{N-k}{N}\, d_y = \frac{N-k}{N} \mathrm{vol}(S_k(x_0))$$

Therefore we have $\dfrac{\mathrm{vol}(S_{k+1}(x_0))}{\mathrm{vol}(S_k(x_0))} = \dfrac{N-k}{k+1}$. $\square$

By the claim ✡, we derive

$$\mathrm{vol}(S_k(x_0)) = \mathrm{vol}(S_0(x_0)) \frac{\mathrm{vol}(S_1(x_0))}{\mathrm{vol}(S_0(x_0))} \cdots \frac{\mathrm{vol}(S_k(x_0))}{\mathrm{vol}(S_{k-1}(x_0))}$$

$$= d_{x_0} \cdot N \cdot \frac{N-1}{2} \cdots \frac{N-k+1}{k} = d_{x_0} \cdot \binom{N}{k}.$$

<u>Theorem 6.12</u>: Let $G=(V,E)$ satisfy $CD(K,\infty)$, $K>0$. Let $x_0 \in V$ and $f_0 := d(x_0,\cdot)$. The following are equivalent:

(1) $\exists\, y_0 \in V$ s.t. $d(x_0,y_0) = \frac{2}{K}$.

(2) $\Gamma_2(f_0) = K\Gamma(f_0)$

(3) $f_0 = \varphi + C$ for a const C and an eigenfunction φ to K of Δ (i.e. $\Delta\varphi + K\varphi = 0$)

(4). G has the hypercube shell structure $HSS(\frac{2}{K}, x_0)$.

Proof: (1) $\Rightarrow$ (2) has been shown in Theorem 6.10.

(2) $\Leftrightarrow$ (3) has been shown in Theorem 6.11.

(3) $\Rightarrow$ (4) By (2)$\Leftrightarrow$(3) and Lemma III.6.6, we know

- $\Gamma(\varphi) = \Gamma(f_0) \equiv$ constant. $\quad (***)$

Notice that at x_0, we have

$$\Gamma(f_0)(x_0) = \frac{1}{2d_{x_0}} \sum_{y \in S_1(x_0)} (f_0(y) - f_0(x_0))^2 = \frac{1}{2}$$

At any vertex x, we have

$$\Gamma(f_0)(x) = \frac{1}{2d_x} \sum_{y \in S_1(x)} (f_0(y) - f_0(x))^2$$

$$= \frac{1}{2d_x} \sum_{\substack{y \in S_1(x) \\ f_0(y) \neq f_0(x)}} 1$$

Now $(***)$ forces $\displaystyle\sum_{\substack{y \in S_1(x) \\ f_0(y) \neq f_0(x)}} 1 = d_x, \quad \forall x \in V$.

$\Rightarrow$ for any x, y with $d(x_0, x) = d(x_0, y)$, we have $x \not\sim y$!

$\Rightarrow$ G is bipartite. (no cycle of odd length).

- It remains to show $p_-^{x_0}(x) = \dfrac{d(x_0, x)}{N} = \dfrac{K\, d(x_0, x)}{2}$.

We consider the equation

$$\Delta\varphi(x) = -K\varphi(x), \quad \forall x \in V$$

$$LHS = \Delta\varphi(x) = \Delta(f_0 - C)(x) = \Delta f_0(x) = \frac{1}{d_x}\sum_{y\in S_1(x)}(f_0(y)-f_0(x))$$

$$= \frac{1}{d_x}\sum_{\substack{y\in S_1(x)\\ f_0(y)<f_0(x)}}(-1) + \frac{1}{d_x}\sum_{\substack{y\in S_1(x)\\ f_0(y)>f_0(x)}} 1$$

$$= p_+^{x_0}(x) - p_-^{x_0}(x).$$

$$RHS = -K\varphi(x) = -K(f_0 - C)(x) = -K\,d(x_0,x) + CK.$$

In particular, at x_0, we have

$$LHS = \Delta\varphi(x_0) = 1 = RHS = CK \Rightarrow C = \frac{1}{K}$$

Then we arrive at

$$p_+^{x_0}(x) - p_-^{x_0}(x) = 1 - K\,d(x_0,x)$$

Moreover, we have

$$p_+^{x_0}(x) + p_-^{x_0}(x) = 1$$

$$\Rightarrow p_-^{x_0}(x) = \frac{K\,d(x_0,x)}{2}.$$

Therefore, we prove that G satisfies $HSS(\frac{2}{K}, x_0)$.

(4) $\Rightarrow$ (1). By $HSS(\frac{2}{K}, x_0)$, we know

$$p_+^{x_0}(x) = 1 - \frac{K\,d(x_0,x)}{2} > 0 \text{ whenever } d(x_0,x) < \frac{2}{K}.$$

That is, for $x\in V$, if $d(x_0,x) < \frac{2}{K}$, then $\exists\, y\in V$ with

$$d(x_0,y) > d(x_0,x).$$

Therefore, there exists $y_0\in V$ s.t. $d(x_0,y_0) = \frac{2}{K}$. This proves (1). $\square$

Combining Theorem 6.10 and Theorem 6.12, we have the following very intriguing consequence.

Corollary III.6.13. Let $G=(V,E)$ satisfy $CD(K,\infty)$, $K>0$.
~~Then~~ Let K be an eigenvalue of Δ of multiplicity r.
Then for any $x_0 \in V$ s.t. $\boxed{d_{x_0} \le r \le d_{min}}$ ~~we have~~ there always exists $y_0 \in V$ s.t. $d(x_0, y_0) = \frac{2}{K} = \mathrm{diam}(G)$.

$d_{x_0} = d_{min}$

~~Next, we discuss a conditions which ensures~~
We can further update Theorem III.6.10 as follows:

Theorem III.6.10' (Sharpened). Let G satisfy $CD(K,\infty)$, $K>0$.

(i) If $\mathrm{diam}(G) = \frac{2}{K}$, then for any $x_0 \in V$ s.t. $\exists y_0 \in V$ satisfying $d(x_0,y_0) = \mathrm{diam}(G)$, G satisfies $HSS(\frac{2}{K}, x_0)$.

(ii) If K is an eigenvalue of multiplicity r, then for any $x_0 \in V$ s.t. $\boxed{d_{x_0} \le r \le d_{min}}$ $\left(\Rightarrow \exists y_0 \in V \text{ s.t. } d(x_0,y_0) = \mathrm{diam}(G)\right)$ G satisfies $HSS(\frac{2}{K}, x_0)$.

$d_{x_0} = d_{min}$

Definition III.6.14: A vertex $x \in V$ is called a pole if there exists a vertex $y \in V$ such that $d(x,y) = \mathrm{diam}(G)$, in which case, y is called an antipole of x (w.r.t. G). A graph G is called self-centered if every vertex is a pole.

- So, if the multiplicity r of the eigenvalue K equals d_{min}, then any $x_0 \in V$ with $d_{x_0} = d_{min}$ is a pole, and $HSS(\frac{2}{K}, x_0)$ holds.
- If we, ~~moreover,~~ assume, the multiplicity $r = d_{max}$, then $d_{max} = r \le d_{min} \Rightarrow G$ is d-regular, and every vertex is a pole. That is, G is self-centered.

Recall our aim Theorem III.6.1. At the moment, we have either (1) or (3) implies $\exists x_0 \in V$ s.t. $HSS(\frac{2}{K}, x_0)$ holds.

(1): diam. sharp d-regular; (3): $\lambda_{d_{max}+1} = K$

Hence, in order to show Theorem III.6.1, it remains to show.

Theorem III.6.15: Let $G=(V,E)$ be a d-regular graph satisfying $CD(K,\infty)$, $K>0$ and $HSS(\frac{2}{K}, x_0)$ for some $x_0 \in V$, then $G \cong \frac{2}{K}$-dim hypercube.

Recall that $HSS(\frac{2}{K}, x_0) \Rightarrow d_x = N = \frac{2}{K}$, $\forall x \in S_1(x_0)$.

Therefore d-regularity implies $d = \frac{2}{K}$. So

Theorem III.6.15' Let $G=(V,E)$ be a d-regular graph satisfying $CD(\frac{2}{d},\infty)$ and $HSS(d, x_0)$ for some $x_0 \in V$.

Then $G \cong$ d-dim. hypercube.

~~Our~~ Small sphere and non-clustering properties.

Definition III.6.16: Let $G=(V,E)$ be a d-regular graph. Let $x \in V$ be a vertex.

(SSP) We say x satisfies the small sphere property (SSP) if
$$\# S_2(x) \leq \binom{d}{2}$$

(NCP) We say x satisfies the non-clustering property (NCP) if, whenever $d_-^x(z) := 2$ holds for all $z \in S_2(x)$, one has that for all $y_1, y_2 \in S_1(x)$, there is at most one $z \in S_2(x)$ satisfying $y_1 \sim z \sim y_2$. $\left(\forall z \in S_2(x),\ d_-^x(z) := \sum_{\substack{y \in S_1(x) \\ y \sim z}} 1\right)$ in-degree.

In order to show Theorem III.6.15', our first step is

Lemma III.6.17: Let $G=(V,E)$ be a d-regular bipartite graph satisfying $CD(\frac{2}{d},\infty)$ at some point $x\in V$. Then x satisfies both (SSP) and (NCP).

For this purpose, we need a reformulation of the Bakry-Émery curvature. Let $K_{G,x}(\infty)$ be the maximal K s.t. $CD(K,\infty)$ holds at x. It is the solution of the following optimization problem.

$$\begin{aligned} &\text{maximize } K \\ &\text{subject to } \Gamma_2(x)-K\Gamma(x)\geq 0. \end{aligned} \qquad (P).$$

Schmuckenschläger observed that the size of these matrices can be reduced by 1: since $\Gamma_2(f)$, $\Gamma(f)$, $\Delta(f)$ all vanish for constant functions f, the $CD(K,N)$ still holds after shifting f by an additive constant. Therefore, we only need consider functions f s.t. $f(x)=0$. Therefore (P) can be reformulated as

$$\begin{aligned} &\text{maximize } K \\ &\text{subject to } \left(\Gamma_2(x)-K\Gamma(x)\right)_{S_1(x)\cup S_2(x),\,S_1(x)\cup S_2(x)}\geq 0. \end{aligned} \qquad (P')$$

Recall that

$$\begin{pmatrix} \Gamma_2(x)_{S_1(x),S_1(x)}-K\Gamma(x)_{S_1(x),S_1(x)} & \Gamma_2(x)_{S_1(x),S_2(x)} \\ \Gamma_2(x)_{S_2(x),S_1(x)}, & \Gamma_2(x)_{S_2(x),S_2(x)} \end{pmatrix}.$$

diagonal

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where $\Gamma(x)_{S_1(x),S_1(x)} = \frac{1}{2}\begin{pmatrix} p_{xy_1} & & \\ & \ddots & \\ & & p_{xy_m} \end{pmatrix} = \cancel{\frac{1}{d_x} I_m = \frac{1}{d} I_m} = \frac{1}{2d_x} I_m$

Lemma III. 6.18 (Schur complement).

Consider a square matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, where M_{11}, M_{22} are square submatrices, and assumpe $M_{22} > 0$. Then the Schur complement

$$M/M_{22} := M_{11} - M_{12} M_{22}^{-1} M_{21} \geq 0 \text{ iff } M \geq 0.$$

This is an linear algebraic lemma obtained from the Gaussian eliminations in solving linear equations.

Let us denote
$$Q(x) = (\Gamma_2(x))_{S_1 \cup S_2, S_1 \cup S_2} / \Gamma_2(x)_{S_2,S_2}$$
$$= \Gamma_2(x)_{S_1,S_1} - \Gamma_2(x)_{S_1,S_2} \Gamma_2(x)_{S_2,S_2}^{-1} \Gamma(x)_{S_2,S_1}.$$

Then (P') can be further reformulated as

$$K_{G,x}(\infty) = \arg\max_K \{ Q(x) - K\Gamma(x)_{S_1,S_1} \geq 0 \}$$

$$\cancel{= \arg\max_K \{ Q(x) - \tfrac{K}{d} I \}}$$

~~Denote the vector~~ $\cancel{v_0(x) = \begin{pmatrix} \sqrt{p_{xy_1}} \\ \sqrt{p_{xy_2}} \\ \vdots \\ \sqrt{p_{xy_m}} \end{pmatrix}}$

~~Then diam~~

$$= \arg\max_K \{ 2d_x Q(x) - K I_m \geq 0 \}$$
$$= \lambda_{min}(2d_x Q(x))$$
$$=: \lambda_{min} A_\infty(x).$$

We call the $\overset{d_x}{\overset{\shortparallel}{m}} \times m$ matrix $A_\infty(x) := 2d_x Q(x)$ the curvature matrix of x. It can be considered as an analogue of the Ricci curvature tensor at x.

This is discovered in Cushing, Kamtue, Liu, Peyerimhoff, Bakry-Émery curvature on graphs as an eigenvalue problem, arXiv 2102.08687

(independently, via diff. method, by Siconolfi, Ricci curvature, graphs and eigenvalues, arXiv: 2102.10134).

Theorem III.6.19. We have the expression of $Q(x)$.

$$Q(x)_{yy} = \frac{1}{2} p_{xy}^2 + \frac{3}{4} p_{xy} p_{yx} - \frac{1}{4} p_{xy} + \frac{3}{4} p_{xy} \sum_{z \in S_2(x)} p_{yz}$$

$$+ \frac{1}{4} \sum_{y' \in S_1(x)} \left(3 p_{xy} p_{yy'} + p_{xy'} p_{y'y}\right) - \sum_{z \in S_2(x)} \frac{p_{xy}^2 p_{yz}^2}{p_{xz}^{(2)}}$$

$$\text{where } p_{xz}^{(2)} = \sum_{y \in S_1(x)} p_{xy} p_{yz} \quad , \quad \forall\, y \in S_1(x)$$

and

$$Q(x)_{y_i y_j} = \frac{1}{2} p_{xy_i} p_{xy_j} - \frac{1}{2} p_{xy_i} p_{y_i y_j} - \frac{1}{2} p_{xy_j} p_{y_j y_i} - \sum_{z \in S_2(x)} \frac{p_{xy_i} p_{y_i z} p_{xy_j} p_{y_j z}}{p_{xz}^{(2)}}.$$

$$\forall\, y_i, y_j \in S_1(x),\ i \neq j$$

Proof. By direct calculation.

~~Proof of Lemma III.6.17.~~

Next, we apply the general theory back to our setting: G is d-regular, bipartite and satisfies $CD(\frac{2}{d}, \infty)$ at x.

Then
$$Q(x)_{y_i, y_j} = \frac{1}{2d^2} - \sum_{\substack{z \in S_2(x) \\ y_i \sim z \sim y_j}} \frac{\frac{1}{d^2} \cdot \frac{1}{d^2}}{\frac{1}{d^2} d_-^x(z)} = \frac{1}{2d^2} - \frac{1}{d^2} \sum_{\substack{z \in S_2(x) \\ y_i \sim z \sim y_j}} \frac{1}{d_-^x(z)}$$

$$Q(x)_{yy} = \frac{1}{2d^2} + \frac{3}{4d^2} - \frac{1}{4d} + \frac{3}{4} \frac{1}{d^2} d_+^x(y) - \sum_{\substack{z \in S_2(x) \\ y \sim z}} \frac{\frac{1}{d^4}}{\frac{1}{d^2} d_-^x(z)}$$

$$= \frac{5 + 3 d_+^x(y)}{4} \cdot \frac{1}{d^2} - \frac{1}{4d} - \frac{1}{d^2} \sum_{\substack{z \in S_2(x) \\ z \sim y}} \frac{1}{d_-^x(z)}$$

$$\Rightarrow \quad A_\infty(x)_{y_i, y_j} = \frac{1}{d} - \frac{2}{d} \sum_{\substack{z \in S_2(x) \\ y_i \sim z \sim y_j}} \frac{1}{d_-^x(z)}$$

$$A_\infty(x)_{yy} = \frac{5 + 3 d_+^x(y)}{2d} - \frac{1}{2} - \frac{2}{d} \sum_{\substack{z \in S_2(x) \\ z \sim y}} \frac{1}{d_-^x(z)}$$

Notice that

$$\sum_{\substack{y_j\in S_1(x)\\ j\neq i}} \frac{2}{d} \sum_{\substack{z\in S_2(x)\\ y_i\sim z\sim y_j}} \frac{1}{d_-^x(z)} = \frac{2}{d} \sum_{\substack{z\in S_2(x)\\ z\sim y_i}} \sum_{\substack{y_j\in S_1(x)\\ j\neq i\\ y_j\sim z}} \frac{1}{d_-^x(z)}$$

$$= \frac{2}{d} \sum_{\substack{z\in S_2(x)\\ z\sim y_i}} \frac{d_-^x(z)-1}{d_-^x(z)} = \frac{2}{d} d_+^x(y_i) - \frac{2}{d} \sum_{\substack{z\in S_2(x)\\ z\sim y_i}} \frac{1}{d_-^x(z)}$$

Therefore :

$$A_\infty(x)_{y_iy_j} = \frac{1}{d} - \frac{2}{d} \sum_{\substack{z\in S_2(x)\\ y_i\sim z\sim y_j}} \frac{1}{d_-^x(z)}$$

and $$A_\infty(x)_{yy} = \left(\frac{2}{d} d_+^x(y) - \frac{2}{d} \sum_{\substack{z\in S_2(x)\\ z\sim y}} \frac{1}{d_-^x(z)}\right) - \frac{1}{2} + \underbrace{\frac{5}{2d}}_{= \frac{1}{d}+\frac{3}{2d}} - \frac{1}{2d} d_+^x(y)$$

$$\Rightarrow \quad A_\infty(x) = -2\Delta_{S_1'(x)} + \frac{1}{d}J + \left(\frac{3}{2d}-\frac{1}{2}\right)I - \frac{1}{2d}\underbrace{d_+^x(y)}_{= d-1}$$

$= d-1$ by bipartiteness and d-regularity.

$$\Rightarrow \quad \boxed{A_\infty(x) = -2\Delta_{S_1'(x)} + \frac{1}{d}J + \frac{2-d}{d} I_d.}$$

$$\Delta_{S_1'(x)} f(y_i) = \frac{1}{d} \sum_{y_j\in S_1(x)} w(y_i,y_j)\,(f(y_j)-f(y_i))$$

where $$w(y_i,y_j) = \sum_{\substack{z\in S_2(x)\\ y_i\sim z\sim y_j}} \frac{1}{d_-^x(z)}$$

Let us denote by $0=\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_d$ the eigenvalues of $\Delta_{S_1'(x)}$ (i.e. $\Delta_{S_1'(x)} f(y) + \lambda_i f(y) = 0$).

Then $$\operatorname{tr}\left(-2\Delta_{S_1'(x)}\right) = 2(\lambda_2+\cdots+\lambda_d)$$

$$\| $$

$$\sum_{y\in S_1(x)} \left(\frac{2(d-1)}{d} - \frac{2}{d}\sum_{\substack{z\in S_2(x)\\ z\sim y}} \frac{1}{d_-^x(z)}\right) = 2(d-1) - \frac{2}{d}\sum_{z\in S_2(x)} \sum_{\substack{y\in S_1(x)\\ y\sim z}} \frac{1}{d_-^x(z)}$$

$$= 2(d-1) - \frac{2}{d}\sum_{z\in S_2(x)} 1 = 2(d-1) - \frac{2}{d}\,\# S_2(x) \qquad \langle 1\rangle$$

On the other hand, $\sigma(A_\infty(x)) = \left\{1+\frac{2-d}{d} = \frac{2}{d},\ 2\lambda_i + \frac{2-d}{d},\ i=2,\dots,d\right\}$

$$CD\left(\tfrac{2}{d},\infty\right) \Rightarrow \lambda_{\min}(A_\infty(x)) \geq \frac{2}{d} \Rightarrow 2\lambda_i + \frac{2-d}{d} \geq \frac{2}{d}$$

$$\Rightarrow \lambda_i \geq \frac{1}{2}. \qquad (*)$$

Therefore, $\operatorname{tr}\left(-2\Delta_{S_1'(x)}\right) = 2\left(\lambda_2 + \cdots + \lambda_d\right) \geq (d-1)$. <2>

Combining <1> and <2> yields

$$(d-1) \leq 2(d-1) - \frac{2}{d}\#S_2(x) \Rightarrow \#S_2(x) \leq \frac{(d-1)d}{2} = \binom{d}{2}.$$

That is, x satisfies (SSP).

Next, we show (NCP). If $d_-^x(z) = 2 \ \forall z \in S_2(x)$, we have

$$2\cdot\#S_2(x) = |E(S_1(x),S_2(x))| = (d-1)\cdot d \Rightarrow \#S_2(x) = \binom{d}{2}.$$

That is, equality holds in the above (SSP) estimate $(*)$. This forces

$$\lambda_2 = \lambda_3 = \cdots = \lambda_d = \frac{1}{2}.$$

That is, $\sigma(-\Delta_{S_1'(x)}) = \{0, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{d-1}\}$

That is, any function vertical to $\mathbb{1}$ is an eigenfct of $-\Delta_{S_1'(x)}$ to $\frac{1}{2}$. Denote $S_1(x) = \{1, 2, \cdots, d\}$.

Consider $f(x) = \delta_k(x) - \delta_\ell(x) = \begin{cases} 0, & x \neq k,\ell \\ 1, & x = k \\ -1. & x = \ell \end{cases}$, $k \neq \ell$.

For any $i \in \{1, 2, \cdots, d\}$, $i \neq k, \ell$.

$$-\Delta_{S_1'(x)}(f)(i) = \frac{1}{2}f(i) = 0$$

$$= \frac{1}{d}\sum_{t}\sum_{\substack{z\in S_2(x)\\ i\sim z\sim t}} \frac{1}{d_-^x(z)}\cdot\left(f(i) - f(t)\right)$$

$$= -\frac{1}{d}\sum_{\substack{z\in S_2(x)\\ i\sim z\sim k}}\frac{1}{d_-^x(z)} + \frac{1}{d}\sum_{\substack{z\in S_2(x)\\ i\sim z\sim \ell}}\frac{1}{d_-^x(z)}$$

$\Rightarrow$ Each off-diagonal entry of $-\Delta_{S_1'(x)}$ equals to the same constant C.

Hence, $\cancel{(D^2-D)C}$ $(d^2-d)C = \operatorname{tr}\left(-\Delta s_i'(x)\right) = \frac{1}{2}(d-1)$ 252

$$\Rightarrow C = \frac{1}{2d}.$$

That is, $\forall\, i \neq k$, $\frac{1}{d} \sum_{\substack{z \in S_2(x) \\ i \sim z \sim k}} \frac{1}{d_-^x(z)} = \frac{1}{2d}$

$$\Rightarrow \frac{1}{2} = \sum_{\substack{z \in S_2(x) \\ i \sim z \sim k}} \frac{1}{d_-^x(z)} = \frac{1}{2} \sum_{\substack{z \in S_2(x) \\ i \sim z \sim k}} 1 \Rightarrow \sum_{\substack{z \in S_2(x) \\ i \sim z \sim k}} 1 = 1.$$

This proves the (NCP) property. $\square$.