Hence, $\cancel{(D^2-D)C}$ $(d^2-d)C = \mathrm{tr}\left(-\Delta s_1^1(x)\right) = \frac{1}{2}(d-1)$

$$\Rightarrow C = \frac{1}{2d}.$$

That is, $\forall\, i \neq k$, $\displaystyle\frac{1}{d}\sum_{\substack{z\in S_2(x)\\ i\sim z\sim k}} \frac{1}{d_-^x(z)} = \frac{1}{2d}$

$$\Rightarrow \frac{1}{2} = \sum_{\substack{z\in S_2(x)\\ i\sim z\sim k}} \frac{1}{d_-^x(z)} \overset{d_-^x(z)=2}{=} \frac{1}{2}\sum_{\substack{z\in S_2(x)\\ i\sim z\sim k}} 1 \Rightarrow \sum_{\substack{z\in S_2(x)\\ i\sim z\sim k}} 1 = 1.$$

This proves the (NCP) property. $\square$

<u>Lemma</u> III 6.17': Let $G=(V,E)$ be a d-regular bipartite graph satisfying $CD(\frac{2}{d},\infty)$ at $x\in V$. Then x satisfies

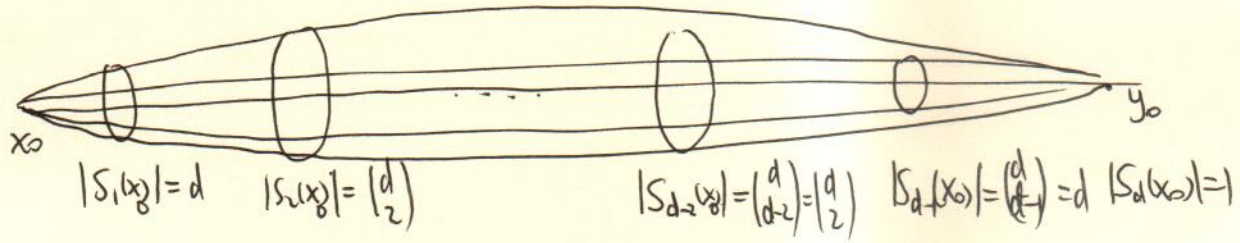
$$\# S_2(x) \leq \binom{d}{2} \qquad \text{(SSP)}$$

Moreover, "=" holds if and only if

for any $y_i, y_k \in S_1(x)$, $i\neq k$. $\displaystyle\sum_{\substack{z\in S_2(x)\\ y_i\sim z\sim y_k}} \frac{1}{d_-^x(z)} = \frac{1}{2}$. (NCP)

<u>Theorem</u> III 6.18. Let $G=(V,E)$ be a d-regular graph satisfying $HSS(d,x_0)$ for some $x_0\in V$, i.e. G is bipartite and $p_-^{x_0}(x) = \frac{d(x_0,x)}{d} = \frac{d_-^{x_0}(x)}{d}$. Suppose (SSP) and (NCP) hold. Then G is isomorphic to the d-dimensional hypercube.

<u>Remark 1</u>: A d-regular graph G satisfying $HSS(d,x_0)$ for $x_0\in V$ has the following structure:



We define an interval between two vertices x, y to be

$$[x,y] = \{z \in V : d(x,z) + d(z,y) = d(x,y)\}$$

Therefore, "for d-regular G with $HSS(d, x_0)$, $\exists!\, y_0 \in V$ s.t. $[x_0, y_0] = V$.

<u>Remark 2</u>: We take the following definition of hypercube graph:

Let $[d] := \{1, 2, \dots, d\}$. Denote the power set by $\mathcal{P}$.

The d-hypercube graph $H^d = (V, E)$ is the graph with

$$V = \mathcal{P}([d]), \quad E = \{\{A, B\} : A, B \in \mathcal{P}([d]),\ \#(A \cup B) \setminus (A \cap B) = 1\}$$

Our aim in Theorem III.6.18 is to establish an isomorphism map between G and H^d. We do this by induction.

<u>Proof of Thm III.6.18</u>:

First, we have an isomorphism

$$\Phi_{\le 1} : B_1(x_0) \to \mathcal{P}_{\le 1}([d]) = \{\emptyset, \{1\}, \dots, \{d\}\}$$

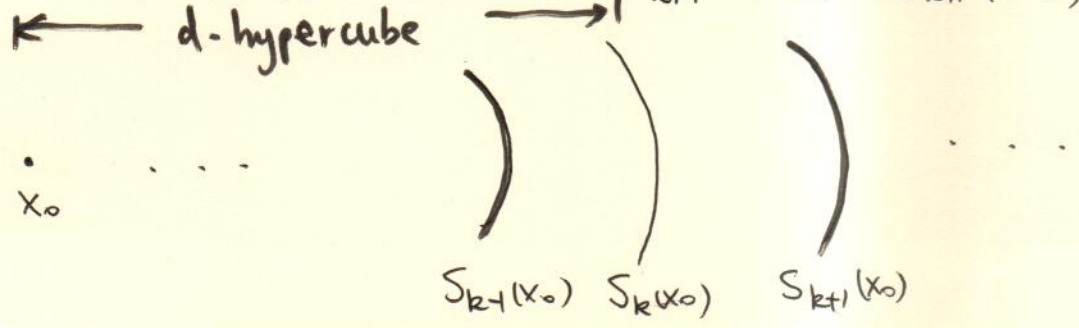
$$x_0 \longmapsto \emptyset$$

$$S_1(x_0) := \{y_1, \dots, y_d\} \quad y_i \longmapsto \{i\}.$$

Observe: $\{\Phi_{\le 1}(x), \Phi_{\le 1}(y)\} \in E(H^d) \Leftrightarrow \{x, y\} \in E(G)$.

Therefore, $\Phi_{\le 1}$ is an isomorphism.

<u>Assume</u> that we have an isomorphism $\Phi_{\le k} : B_k(x_0) \to \mathcal{P}_{\le k}([d])$ for some $k \ge 1$. We next show $B_{k+1}(x_0) \cong \mathcal{P}_{\le k+1}([d])$



Let us consider the bipartite graph with vertex set $S_{k-1}(x_0) \sqcup S_{k+1}(x_0)$, and edge set

$$R = \{\{x, y\} : x \in S_{k-1}(x_0),\ y \in S_{k+1}(x_0) \text{ s.t. } d(x,y) = 2\}$$
$$\text{or } x \in S_{k+1}(x_0),\ y \in S_{k-1}(x_0).$$

We denote the degree of each vertex in this bipartite graph by $\deg_R(\cdot)$.

Then we have

$$(\star) \quad \boxed{\sum_{x \in S_{k-1}(x_0)} \deg_R(x) = |R(S_{k-1}(x_0), S_{k+1}(x_0))| = \sum_{z \in S_{k+1}(x_0)} \deg_R(z)}$$

<u>Claim 1</u>: (SSP) $\Rightarrow \deg_R(x) \leq \binom{d-k+1}{2}$, $\forall x \in S_{k-1}(x_0)$.

with "=" holds iff (NCP).

<u>Claim 2</u>: $d_-^{x_0}(z) = d(x_0, z)$, $z \in S_{k+1}(x_0) \Rightarrow \deg_R(z) \geq \binom{k+1}{2}$, $\forall z \in S_{k+1}(x_0)$

with "=" holds iff $\# \bigcup_{\substack{y \sim z \\ y \in S_k(x_0)}} \Phi_{\leq k}(y) = k+1$.

number of elements in the set.

<u>Proof of Claim 1</u>: Observe that for $x \in S_{k-1}(x)$

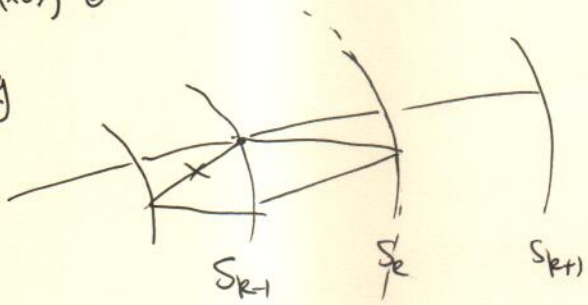
① $\deg_R(x) = \# S_2(x) \cap S_{k+1}(x_0)$, ② $S_2(x) = (S_2(x) \cap S_{k+1}(x_0)) \sqcup (S_2(x) \cap B_k(x_0))$

③ (SSP) $\Rightarrow \# S_2(x) \leq \binom{d}{2}$.

④ $B_k(x_0) \cong \mathcal{P}_{\leq k}([d]) \Rightarrow \#(S_2(x) \cap B_k(x_0))$

$= \binom{d}{2} - \#$ set of $A \in \mathcal{P}_{k+1}([d])$ containing $\Phi_{\leq k}(x) \in \mathcal{P}_{k-1}([d])$ ~~which has can~~

$= \binom{d}{2} - \binom{d-(k-1)}{2}$.

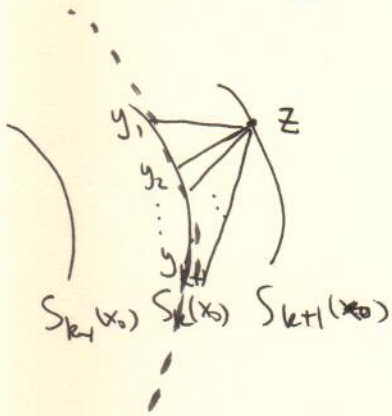


Combining the above ① ② ③ ④, we derive

$$\deg_R(x) = \# S_2(x) - \# |S_2(x) \cap B_k(x_0)|$$
$$\overset{③}{\leq} \binom{d}{2} - \left[\binom{d}{2} - \binom{d-(k-1)}{2} \right] = \binom{d-k+1}{2}.$$

"=" holds iff $\# (S_2(x)) = \binom{d}{2}$, by Lemma III 6.17', this is equivalent to (NCP).

This proves Claim 1. □



<u>Proof of Claim 2</u>:

$$d_-^{x_0}(z) = k+1, \ \forall z \in S_{k+1}(x_0).$$

Let $y_1, \cdots, y_{k+1} \in S_k(x_0)$ be the vertices in $S_k(x_0)$ which are adjacent to z.

Denote $A_i := \Phi_{\leq k}(y_i) \in \mathcal{P}_k([d])$, $i = 1, 2, \cdots, k+1$.

~~Then~~ For any $x \in S_{k-1}(x_0)$, $x \sim y_i$, we have

$$\Phi_{\leq k}(x) \in \mathcal{P}_{k-1}([d]), \text{ and } \Phi_{\leq k}(x) \subset \Phi_{\leq k}(y_i) \text{ is}$$

a subset of $k-1$ element.

Therefore $\deg_R(z) = \# \bigcup_{i=1}^{k+1} \mathcal{P}_{k-1}(A_i)$

Claim 2 follows directly from the <u>Lemma III.6.19</u> below.

<u>Lemma III. 6.19</u>: Let $k, d \in \mathbb{N}$ with $k < d$. Let $A_1, \cdots, A_{k+1}$ be pairwise distinct k-element subsets of $[d] = \{1, 2, \cdots, d\}$, i.e. $A_i \in \mathcal{P}_k([d])$, $A_i \neq A_j$, $\forall i, j$, $i \neq j$.

Then
$$\# \bigcup_{i=1}^{k+1} \mathcal{P}_{k-1}(A_i) \geq \binom{k+1}{2}$$

Moreover, equality holds iff $\# \bigcup_{i=1}^{k+1} A_i = k+1$.

<u>Proof</u>: Intuitively, the smallest number is attained when $\# \bigcup_{i=1}^{k+1} A_i = k+1$, that is, any two sets A_i, A_j has at largest intersection as possible. In this case, we have the number of $(k-1)$-element subsets of A_i's is simply $\binom{k+1}{2}$.

We first observe that $\# A_i \cap A_j \le k-1$ (Since $A_i \ne A_j$)

and $\# (A_i \cup A_j) \ge k+1$ for $i \ne j$.

We calculate for any $j \in \{0, \cdots, k\}$.

$$\# \bigcup_{i=1}^{j+1} \mathcal{P}_{k-1}(A_i) = \# \bigcup_{i=1}^{j} \mathcal{P}_{k-1}(A_i) + \# \mathcal{P}_{k-1}(A_{j+1}) - \# \left(\mathcal{P}_{k-1}(A_{j+1}) \cap \bigcup_{i=1}^{j} \mathcal{P}_{k-1}(A_i) \right)$$

$$= \# \bigcup_{i=1}^{j} \mathcal{P}_{k-1}(A_i) + k - \# \left(\mathcal{P}_{k-1}(A_{j+1}) \cap \bigcup_{i=1}^{j} \mathcal{P}_{k-1}(A_i) \right)$$

$$\text{(i)} \quad \ge \# \bigcup_{i=1}^{j} \mathcal{P}_{k-1}(A_i) + k - \sum_{i=1}^{j} \# \left(\mathcal{P}_{k-1}(A_{j+1}) \cap \mathcal{P}_{k-1}(A_i) \right)$$

$$\text{(ii)} \quad \ge \# \bigcup_{i=1}^{j} \mathcal{P}_{k-1}(A_i) + k - j$$

We use if $B \in \mathcal{P}_{k-1}(A_j) \cap \mathcal{P}_{k-1}(A_i)$, then $B \subset A_i \cap A_j$

$$\Rightarrow \underset{k-1}{\underset{\shortparallel}{\# B}} \le \# (A_i \cap A_j) \le k-1 \Rightarrow B = A_i \cap A_j$$

$$\Rightarrow \# \mathcal{P}_{k-1}(A_j) \cap \mathcal{P}_{k-1}(A_i) \le 1. \quad \forall i \ne j.$$

Therefore, $\# \bigcup_{i=1}^{k+1} \mathcal{P}_{k-1}(A_i) \ge \# \bigcup_{i=1}^{k} \mathcal{P}_{k-1}(A_i) \ge \# \bigcup_{i=1}^{k-1} \mathcal{P}_{k-1}(A_i) + 1 \ge \cdots$

$$\ge 1 + 2 + \cdots + k = \binom{k+1}{2}.$$

<u>Sharpness</u> $\Rightarrow$ "=" holds at (i) & (ii).

$k=1$. trivial

$k \ge 2$: $j=1$. (ii) "=" $\Rightarrow \# (A_1 \cap A_2) = k-1 \Rightarrow \# (A_1 \cup A_2) = k+1$.

$k \ge 3$. $j=2$. (ii) "=" $\Rightarrow \# (A_1 \cap A_3) = \# (A_2 \cap A_3) = k-1$.

(i) "=" $\Rightarrow A_1 \cap A_3 \ne A_2 \cap A_3$.

Therefore, $\#(A_1 \cup A_2) \cap A_3 = \#((A_1 \cap A_3) \cup (A_2 \cap A_3)) \geq k = \# A_3$

$\Rightarrow \quad A_3 \subset A_1 \cup A_2$, where $\#(A_1 \cup A_2) = k+1$.

Reordering A_i, yields $A_i \subset A_1 \cup A_2$, $\forall i$.

Hence $\# \bigcup_{i=1}^{k+1} A_i = k+1$. as desired. $\square$.

Combining ⊛ and Claims 1 & 2 on pp. 254, we have

$$\# S_{k-1}(x_0) \cdot \binom{d-k+1}{2} \geq |R(S_{k-1}(x_0), S_{k+1}(x_0))| \geq \# S_{k+1}(x_0) \cdot \binom{k+1}{2}$$

$$\binom{d}{k-1}\binom{d-k+1}{2} \overset{!!!}{=} \binom{d}{k+1}\binom{k+1}{2}$$

Therefore, "=" holds in Claims 1 & 2 everywhere.

In particular, we have

$$\# \bigcup_{i=1}^{k+1} \Phi_{\leq k}(y_i) = k+1.$$

We can then define

$$\Phi_{k+1} : S_{k+1}(x_0) \longrightarrow \mathcal{P}_{k+1}([d])$$
$$z \longmapsto \bigcup_{i=1}^{k+1} \Phi_{\leq k}(y_i), \quad \text{where } y_1, \cdots, y_{k+1} \in S_k(x_0),\ y_i \sim z.$$

Then $\Phi_{\leq k}(y_i) \in \mathcal{P}_k([d])$ are exactly the k-elements subsets of $\Phi_{k+1}(z) \in \mathcal{P}_{k+1}([d])$.

Therefore $\quad y \sim z \iff \Phi_{\leq k}(y) \sim \Phi_{k+1}(z)$.

for any $y \in S_k(x_0)$, $z \in S_{k+1}(x_0)$.

We define $\Phi_{\leq k+1} : B_{k+1}(x_0) \longrightarrow \mathcal{P}_{k+1}([d])$ via

$$x \longmapsto \begin{cases} \Phi_{k+1}(x) : & \text{if } x \in S_{k+1}(x_0) \\ \Phi_{\leq k}(x) & \text{if } x \in B_k(x_0) \end{cases}$$

We have $x \sim y \iff \Phi_{\leq k+1}(x) \sim \Phi_{\leq k+1}(y)$.

Since $\# B_{k+1}(x_0) = \# P_{k+1}([d])$, we only need to show that $\Phi_{\leq k+1}$ is injective.

Suppose there are $z_1, z_2 \in S_{k+1}(x_0)$, $z_1 \neq z_2$, that

$$(*) \quad \Phi_{\leq k+1}(z_1) = \Phi_{\leq k+1}(z_2) = Z \in P_{k+1}([d]).$$

Let $x \in S_{k-1}(x_0)$ s.t. $x \sim y \sim z$ for some $y \in S_k(x_0)$.

Let $X = \Phi_{\leq k+1}(x)$, then $X \in P_{k-1}([d])$ and $X \subset Z$.

There are only two Y s.t. $X \subset Y \subset Z$, $Y \in P_k([d])$.

That is, $d_-^X(z) = 2$, $\forall z \in S_{k+1}(x_0) \cap S_2(x)$.

Since $B_k(x_0)$ is isomorphic to hypercube, we have

$$d_-^X(z) = 2, \quad \forall z \in S_2(x_0)$$

Then (NCP) $\Rightarrow$ $\forall\, y_1, y_2 \in S_1(x)$, there exists at most one $z \in S_2(x)$ s.t. $y_1 \sim z \sim y_2$.

However $(*)$ tells $y_1 \sim z_1 \sim y_2$, $y_1 \sim z_2 \sim y_2$. contradicting to (NCP).

Therefore $\Phi_{\leq k+1}$ is injective. $\square$.