This immediately implies that $D_n \to \infty$ as $n \to \infty$. $\square$

<u>Remark</u>. In fact, Noga Alon conjectured that "almost all" d-regular graphs G on n vertices satisfies

$$\lambda_2(G) \geq 1 - \frac{2\sqrt{d-1}}{d} - o(1) \quad \text{as } n \to \infty.$$

[N. Alon] Eigenvalues and expanders, Combinatorica 6 (1986), no. 2 83–96.

Of course, "almost all" should be interpreted as probability $\to 1$ as $n \to \infty$ according to various models of a random graph. This conjecture was confirmed by Joel Friedman.

[J. Friedman] A proof of Alon's second eigenvalue conjecture and related problems, Mem. Amer. Math. Soc. 195 (2008), no. 910, viii + 100 pp.

<u>Example</u>: Eigenvalues of $\mathbb{Z}_n$ (n-cycle). $V = \{0, 1, \cdots, n-1\}$

$$E = \{\{0,1\}, \{1,2\}, \cdots, \{n, n-1\}, \{n-1, 0\}\}$$
$$= \{\{k, k+1\}, k = 0, 1, \cdots, n-1\}.$$

The equation $\Delta f(k) + \lambda f(k) = 0$ can be written as

$$\frac{1}{2}(f(k+1) + f(k-1)) - f(k) + \lambda f(k) = 0$$

i.e. $(*)$ $f(k+1) + f(k-1) - 2(1-\lambda) f(k) = 0$, $k = 0, 1, 2, \cdots, n-1$.

We know $\lambda = 0$ is always a simple eigenvalue, and $\lambda = 2$ is an eigenvalue (simple) iff $\mathbb{Z}_n$ is bipartite, i.e. n even. So we can assume $\lambda \in (0, 2)$ in the following.

Let us consider the equation $(*)$ on the whole $\mathbb{Z}$.

It's solutions form a linear space and are uniquely determined once we know the values of $f(0)$ and $f(1)$. Hence, the solution space is a 2-dim linear space.

Let us consider a specific form:

$$f(k) = r^k, \quad r \in \mathbb{C}.$$

Then (*) tells

$$r^{k+1} + r^{k-1} - 2(1-\lambda) r^k = 0$$

$$\Rightarrow \quad r^2 - 2(1-\lambda) r + 1 = 0$$

$$\Rightarrow \quad r = \frac{2(1-\lambda) \pm \sqrt{4(1-\lambda)^2 - 4}}{2} = (1-\lambda) \pm i\sqrt{1-(1-\lambda)^2}$$

$$= e^{\pm i\theta}$$

where $\theta \in (0, \pi)$ is determined by

$$\cos\theta = 1 - \lambda \quad \left(\sin\theta = \sqrt{1-(1-\lambda)^2} \right)$$

Hence, we have 2 independent complex valued solutions

$$f_1(k) = e^{ik\theta}, \quad f_2(k) = e^{-ik\theta}.$$

By Euler formula, we derive the following 2 indep. real solutions

$$f_1(k) = \cos k\theta, \quad f_2(k) = \sin k\theta. \qquad (*)$$

Now, we impose the n-periodicity to make functions on $\mathbb{Z}_n$.

i.e. $\quad f(k+n) = f(k)$.

The fcts (*) are n-periodic iff $n\theta$ is a multiple of 2π,

i.e.

$$\theta = \frac{2\pi l}{n}.$$

Recall $\theta \in (0, \pi)$, we have $l \in (0, \frac{n}{2})$.

For each $l \in (0, \frac{n}{2}) \cap \mathbb{Z}$, we obtain an eigenvalue $\lambda = 1 - \cos\frac{2\pi l}{n}$.

of multiplicity 2. (with $f_1(k) = \cos\frac{2\pi k l}{n}$, $f_2(k) = \sin\frac{2\pi k l}{n}$ as eigenfunctions).

(**) $\begin{cases} \text{If } n \text{ is odd, we have } \lambda = 1 - \cos\frac{2\pi l}{n},\ l = 1, 2, \cdots, \frac{n-1}{2} \\ \text{and } \lambda_1 = 0 \text{ (simple)}. \text{ Total counting with multiplicity} = n-1+1 = n. \\ \text{If } n \text{ is even, we have } \lambda = 1 - \cos\frac{2\pi l}{n},\ l = 1, 2, \cdots, \frac{n}{2} - 1 \\ \text{and } \lambda_1 = 0,\ \lambda_n = 2 \text{ (simple)}. \text{ Total counting with multiplicity} \\ = 2n - 2 + 2 = n. \end{cases}$

Therefore, in both cases, we have found all the eigenvalues.

<u>Lemma</u>. All the Laplacian eigenvalues of the n-cycle graph $\mathbb{Z}_n$ with multiplicity are given by the sequences

$$1 - \cos\frac{2\pi j}{n}, \quad j = 0, 1, \cdots, n-1.$$

<u>Proof</u>. This is a reformulation of (**). $\square$

We can check the estimate $\lambda_2 \geq \frac{1}{D\,\mathrm{vol}(G)}$ for $G = \mathbb{Z}_n$.

The diameter of $\mathbb{Z}_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ \frac{n-1}{2} & \text{if } n \text{ odd.} \end{cases} = \left\lfloor \frac{n}{2} \right\rfloor$

The volume of $\mathbb{Z}_n = 2n$

Hence $\lambda_2 \geq \frac{1}{D\,\mathrm{vol}(G)} = \frac{1}{\left\lfloor \frac{n}{2} \right\rfloor \cdot 2n} \geq \frac{1}{n^2}$.

Meanwhile we know $\lambda_2 = 1 - \cos\frac{2\pi}{n} \approx \frac{2\pi^2}{n^2}$ (by Taylor) when n is large enough.

This show the estimate works good.

However, there are many cases in which the estimate is far from sharp. One example is given by the <u>Cartesian product</u> of cycles ("discrete tori")

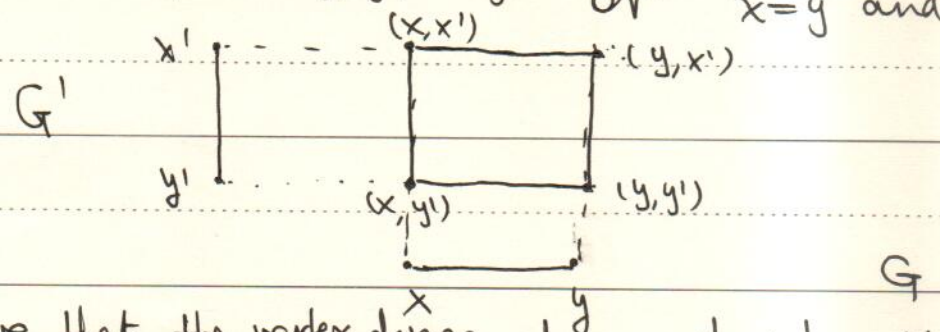
<u>Definition</u>. The Cartesian product of two graphs $G=(V,E)$ and $G'=(V',E')$ is a graph

$$G \times G' = (V \times V', E^{G\times G'})$$

where the edge set is defined by

$\cancel{(x,y)\sim(x',y)}$

$(x,x') \sim (y,y')$ if either $x\sim y$ in E and $x'=y'$ or $x=y$ and $x'\sim y'$ in E'



<u>Observe</u> that the vertex degree $d_{(x,x')} = d_x + d_{x'}$, $\forall x\in V$, $x'\in V'$.

Next, we explore how the eigenvalues (and eigenfunctions) ~~relate~~ of the product graph $G\times G'$ relate to that of the individual graphs G and G'.

Let $f: V\to\mathbb{R}$ and $\alpha\in\mathbb{R}$ be s.t.

$$\Delta^G f + \alpha f = 0$$

Similarly, $g: V'\to\mathbb{R}$ and $\beta\in\mathbb{R}$ satisfy

$$\Delta^{G'} g + \beta g = 0$$

Define the function $h: V\times V'\to\mathbb{R}$ via

$$h(x,x') := f(x)\,g(x'), \quad \forall x\in V,\ x'\in V'.$$

We can calculate

$$(\Delta^{G\times G'} h)((x,x')) = \cancel{\frac{1}{d_{(x,x')}}} \frac{1}{d_{(x,x')}} \sum_{\substack{(y,y')\\ \{(x,x'),(y,y')\}\in E^{G\times G'}}} \big(h(y,y') - h(x,x')\big)$$

$$= \frac{1}{d_x + d_{x'}}\left(\sum_{\substack{y\in V\\ \{x,y\}\in E}} \big(h(y,x') - h(x,x')\big) + \sum_{\substack{y'\in V'\\ \{x',y'\}\in E'}} \big(h(x,y') - h(x,x')\big)\right)$$

$$= \frac{d_x}{d_x+d_{x'}} \cdot \frac{1}{d_x} \sum_{y:\{x,y\}\in E} (f(y)-f(x))\, g(x') + \frac{d_{x'}}{d_x+d_{x'}} \cdot \frac{1}{d_{x'}} \sum_{y':\{x',y'\}\in E'} f(x)\,(g(y')-g(x'))$$

$$= -\frac{d_x\alpha + d_{x'}\beta}{d_x+d_{x'}}\, f(x)\, g(x')$$

$$= -\frac{d_x\alpha + d_{x'}\beta}{d_x+d_{x'}}\, h(x,x'). \qquad (***)$$

In particular, we see when $d_x \equiv \text{const}$, $d_{x'} \equiv \text{const}$, that is, when G, G' are regular graphs, h is an eigenfunction of $G \times G'$!

<u>Theorem</u>: Let $G=(V,E)$ ($|V|>1$) be a d-regular graph, $G'=(V',E')$ ($|V'|>1$) be a d'-regular graph. Let $\{\alpha_k\}_{k=1}^{|V|}$ and $\{\beta_\ell\}_{\ell=1}^{|V'|}$ be the sequences of eigenvalues of G and G' respectively, counting with multiplicities. Then the ~~sequence of~~ eigenvalues of the product $G \times G'$ are given by

$$\left\{ \frac{d\alpha_k + d'\beta_\ell}{d+d'},\ k=1,\dots,|V|,\ \ell=1,\dots,|V'| \right\}.$$

<u>Proof</u>. Let $\{f_k\}_{k=1}^{|V|}$ be a basis of the space of functions on V with $\Delta^G f_k + \alpha_k f_k = 0$. Let $\{g_\ell\}_{\ell=1}^{|V'|}$ be a basis of the functions on V' with $\Delta^{G'} g_\ell + \beta_\ell g_\ell = 0$.

From the calculation in $(***)$, we know the function

$$h_{k\ell} : V \times V' \to \mathbb{R} : h_{k\ell}(x,x') := f_k(x)\, g_\ell(x')$$

is the eigenfunction with the eigenvalue $\frac{d\alpha_k + d'\beta_\ell}{d+d'}$.

Observe that $|V \times V'| = |V|\,|V'|$, it remains to show $\{h_{k\ell}\}$ is a linearly independent sequence of functions on $V \times V'$. Since then, we can conclude that $\left\{\frac{d\alpha_k + d'\beta_\ell}{d+d'}\right\}$ exhausts all the eigenvalues.

of $G \times G'$.

<u>Claim</u>. $\{h_{kl}\}$ is a linearly independent sequence.

<u>Proof</u>. Let $a_{kl} \in \mathbb{R}$, $k=1,\cdots,|V|$, $l=1,\cdots,|V'|$, satisfy

$$\sum_{k,l} a_{kl} h_{kl}(x,x') = 0, \quad \forall (x,x') \in V \times V'.$$

Then $\sum_{k,l} a_{kl} f_k(x) g_l(x') = 0$, $\forall x \in V$, $\forall x' \in V'$.

Therefore, for any given $x' \in V'$, we have

$$\sum_k \left(\sum_l a_{kl} g_l(x') \right) f_k(x) = 0, \quad \forall x \in V.$$

Recall $\{f_k\}$ is linearly indep., we get

$$\sum_l a_{kl} g_l(x') = 0, \quad \forall x' \in V', \ \forall k$$

Therefore, $a_{kl} = 0$, $\forall k, \forall l$. $\square$

<u>Corollary</u>. Let $G = (V,E)$ be a regular graph with $|V|>1$. We denote $G^m := \underbrace{G \times G \times \cdots \times G}_{m \text{ times}}$.

Then the eigenvalues of G^m are given by

$$\left\{ \frac{\alpha_{k_1} + \alpha_{k_2} + \cdots + \alpha_{k_m}}{m} : k_i = 1, \cdots, |V|, \ i = 1, \cdots, m \right\}$$

~~for all~~ where $\{\alpha_k. \ k=1,\cdots,|V|\}$ is the sequence of eigenvalues of G.

<u>Proof</u>. This follows an induction argument in m.

$m=1$: nothing ~~not~~ to show.

$m=2$: by the Theorem.

Suppose the coro. has been shown for G^m. Then for G^{m+1}, we have $G^{m+1} = G^m \times G$. By the Theorem, its eigenvalues are

given by $\frac{md}{md+d}\frac{\alpha_{k_1}+\cdots+\alpha_{k_m}}{m} + \frac{d}{md+d}\alpha_{k_{m+1}} = \frac{d(\alpha_{k_1}+\cdots+\alpha_{k_{m+1}})}{(m+1)d}$. $\square$

Back to the example of $\mathbb{Z}_n$. Consider $(\mathbb{Z}_n)^m$. We immediately know its eigenvalue

$$\lambda_2 = \frac{1-\cos\frac{2\pi}{n}+0\cdots+0}{m} = \frac{1-\cos\frac{2\pi}{n}}{m}$$

$$\approx \frac{2\pi^2}{m\cdot n^2} \quad \text{when } n \text{ is large enough.}$$

On the other hand, $\mathrm{vol}((\mathbb{Z}_n)^m) = 2m\cdot n^m$

and $\mathrm{diam}((\mathbb{Z}_n)^m) \sim nm$

The estimate

$$\lambda_2 \geq \frac{1}{D\,\mathrm{vol}(G)} \sim \frac{1}{2m^2 n^{m+1}}$$

only works good for $m=1$.

Another interesting example one can develop via the construction of taking Cartesian products is a hypercube. Take the "simplest" graph K_2 $\bullet\!\!-\!\!\bullet$ as a start, an n-dimensional hypercube $H^n := (K_2)^n$.

H^n is a regular graph with vertex degree n. Therefore, $\mathrm{vol}(H^n) = n\cdot 2^n$.

Another interesting observation is $\mathrm{diameter}(H^n) = n$. This can be seen as follows: label K_2 as $0\!\!-\!\!1$. Then any vertex of H^n can be labeled by a n-digit coordinate $a_1a_2\cdots a_n$, where $a_i \in \{0,1\}$, $i=1,\cdots,n$. By the rule of Cartesian products, one see that $a_1a_2\cdots a_n$ and $b_1b_2\cdots b_n$ are adjacent iff the two coordinates differ only in 1 digit. Then it is direct

to see the largest distance between two vertices is the distance between $a_1 a_2 \cdots a_n$ and $(1-a_1)(1-a_2)\cdots(1-a_n)$. The distance is of course equal to n.

Since K_2 is connected and bipartite, its eigenvalues are $\alpha_1 = 0$ and $\alpha_2 = 2$. Therefore the eigenvalues of H^n are

$$\frac{\alpha_{k_1} + \alpha_{k_2} + \cdots + \alpha_{k_n}}{n}, \quad k_i \in \{1, 2\}, \ i = 1, \cdots, n.$$

By a counting argument, we obtain the eigenvalues are

0	with multiplicity	$1 := \binom{n}{0}$
$\frac{2}{n}$		$n = \binom{n}{1}$
$\frac{4}{n}$		$\frac{n(n-1)}{2} = \binom{n}{2}$
$\vdots$		$\vdots$
$\frac{2(n-1)}{n}$		$n = \binom{n}{n-1}$
2		$1 = \binom{n}{n}$
$\frac{2j}{n}, j = 0, 1, \cdots, n$		$\binom{n}{j}$

The appearance of 2 tells us H^n is bipartite.

In particular, we have $\lambda_2 = \frac{2}{n} = \frac{2}{D}$

So the estimate $\lambda_2 \geq \frac{1}{\mathrm{Diam}(G)}$ does not work well for H^n.

Laplacian eigenfunctions

Next we transfer our attension to the eigenfunctions of a graph, rather than its eigenvalues.

For finite connected (simple) graphs, the first eigenfct f_1 to $\lambda_1 = 0$ is simply a non zero constant function.

For the second eigenfunction f_2 to $\lambda_2 > \lambda_1 = 0$, we know $0 = (f_2, 1) = \sum_{u \in V} f_2(u)\,du$, $f_2 \not\equiv 0$. Therefore, there must exists $u, v \in V$ s.t. $f_2(u) f_2(v) < 0$. So a very natural question is: Are the induced graphs of $\{u \in V : f_2(u) > 0\}$ and $\{v \in V : f_2(v) < 0\}$ connected?

Recall in our discussion of examples, we derive eigenfcts in the process of finding out all eigenvalues. Let us first check some examples:

<u>Example 1</u>: n-cycle $\mathbb{Z}_n$

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1 - \cos\frac{2\pi}{n}, \quad \lambda_4 = \lambda_5 = 1 - \cos\frac{4\pi}{n} \quad \cdots$$

$$f_2(k) = \cos\frac{2\pi k}{n} \qquad f_4(k) = \cos\frac{4\pi k}{n}$$

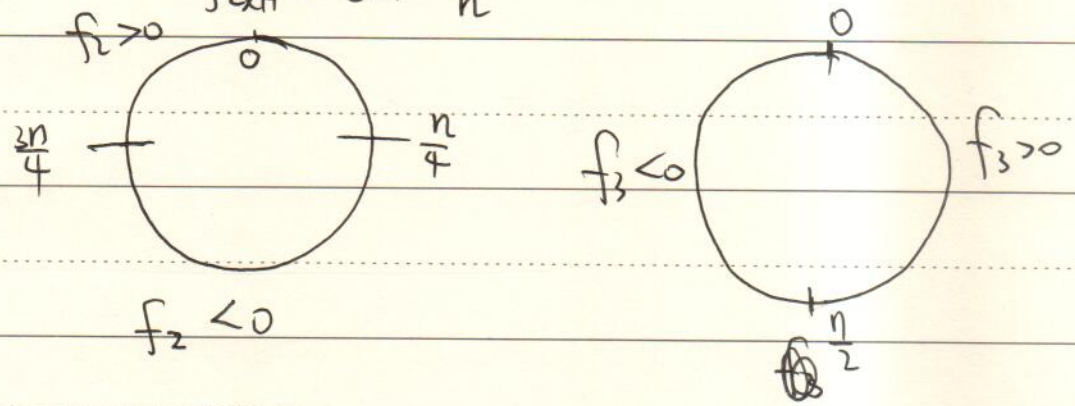
$$k = \{0, 1, \cdots, n-1\}$$

$$f_3(k) = \sin\frac{2\pi k}{n} \qquad f_5(k) = \sin\frac{4\pi k}{n}$$

$$\lambda_{2\ell} = \lambda_{2\ell+1} = 1 - \cos\frac{2\pi\ell}{n} \quad \cdots$$

$$f_{2\ell}(k) = \cos\frac{2\pi k\ell}{n}$$

$$f_{2\ell+1} = \sin\frac{2\pi k\ell}{n}$$



Both are connected!

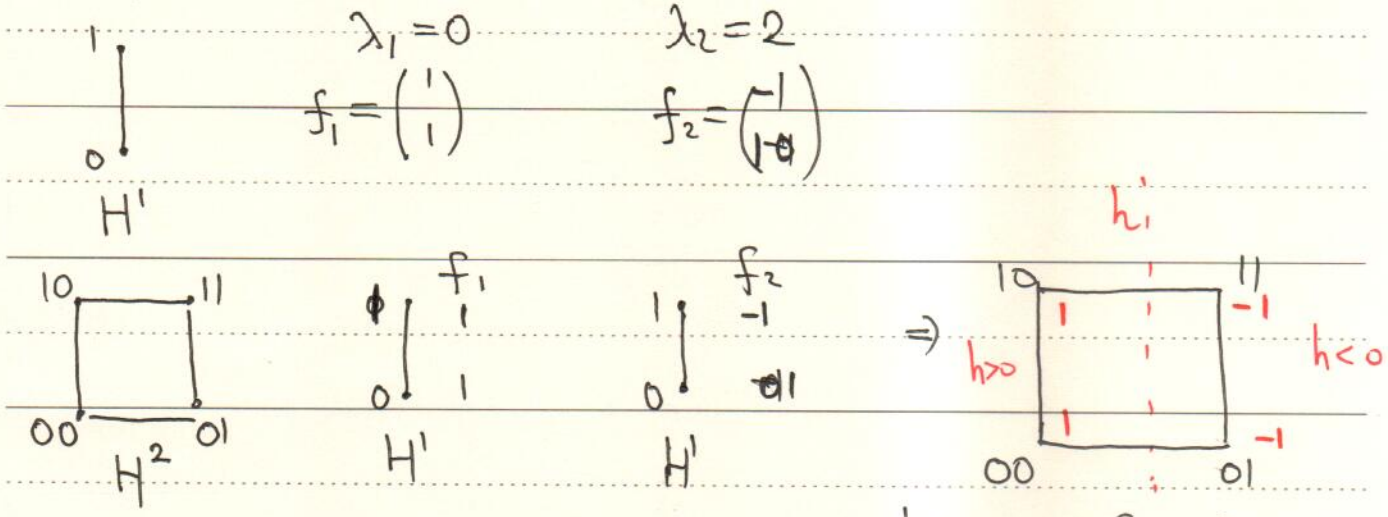
When n large enough, $\underbrace{\{u : f_{2\ell}(u) > 0\} \text{ and } \{v : f_{2\ell}(v) < 0\}}_{\{u : f_{2\ell}(u) \neq 0\}}$

has $2l$ connected component.

<u>Example 2</u>: Hypercubes H^n, $n=1,2,\ldots$

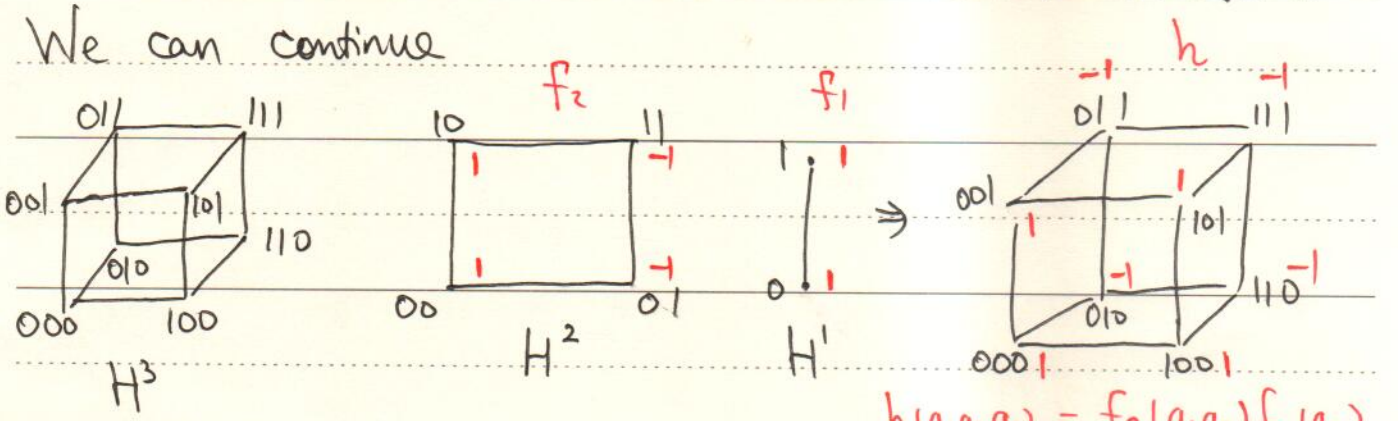
$\lambda_1 = 0$ $\qquad$ $\lambda_2 = 2$

$f_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $f_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$h(a_1 a_2) = f_1(a_1) f_2(a_2)$

We can continue



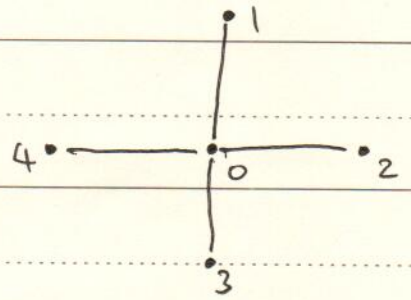
$h(a_1 a_2 a_3) = f_2(a_1 a_2) f_1(a_3)$

In the context of manifolds, "nodes" are defined as points where the eigenfunction f vanishes, i.e. $\{x \in M : f(x) = 0\}$. The <u>nodal sets</u> $\{x \in M : f(x) = 0\}$ are known to be of zero ~~[illegible]~~ measure. The term "nodal domain" refers to the connected components of the set $\{x \in M : f(x) \neq 0\}$. Courant's nodal domain theorem (R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. 1, Interscience, New York, 1953, Chap. 6, §6) is stated as: "Given the self-adjoint 2nd order differential equation $Lu + \lambda \rho u = 0$ $(\rho > 0)$ for a domain G with arbitrary homogeneous

boundary conditions; if its eigenfunctions are ordered according to increasing eigenvalues, then the nodes of the n-th eigenfunction u_n divide the domain into <u>no more than n</u> subdomains." Courant's "subdomains" have since then become known as <u>nodal domains</u>.

Examples 1 and 2 suggest that an analogue discrete version of Courant's nodal domain theorem still holds. However, the following examples, showing the ~~subtlity~~ subtilty of the discrete situation, present the main challenge of establishing a discrete nodal domain theorem.

<u>Example 3</u>. Star graph. (given by Friedman: Some geometric aspect of graphs and their eigenfunctions, Duke Math. J., 69 : 487–525, 1993).



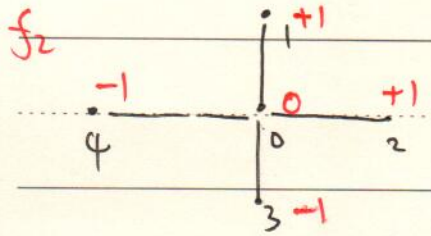
Its Laplacian Matrix.

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & -1/4 & -1/4 & -1/4 & -1/4 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 0 & 1 \end{array}$$

Its eigenvalues are

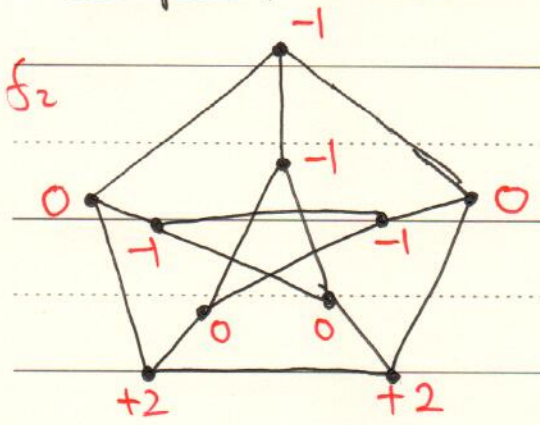
$$0,\ 1,\ 1,\ 1,\ 2$$

An eigenfunction f_2 to the eigenvalue 1 can be



In this case, $\{u : f_2(u) \neq 0\}$ has 4 connected components !!

Example 4. Petersen graph. (10 vertices, const. vertex degree 3)



$\lambda_1 = 0$

$\lambda_2 = \frac{2}{3}$ with multiplicity 5.

f_2

$\{u : f_2 \neq 0\}$ has 3 connected components.

The subtilty of discrete spaces in contrast to their continuous analogue, is that a function can jump from being positive directly to being negative. It is not necessary to pass through a vertex with zero value. This reminds us to distinguish 2 kinds of "nodal domains".

Definition. A ~~strong~~ positive (negative) strong nodal domain of a function $f : V \to \mathbb{R}$ is a maximal connected induced subgraph of G on vertices $\{u \in V : f(u) > 0\}$.
($\{u \in V : f(u) < 0\}$)

A positive (negative) weak nodal domain of a function $f : V \to \mathbb{R}$ is a maximal connected induced subgraph of G on vertices $\{u \in V : f(u) \geq 0\}$ ($\{u \in V : f(u) \leq 0\}$) that contains at least one nonzero vertex. (i.e., u s.t. $f(u) \neq 0$).

We denote by $\mathfrak{S}(f)$ and $\mathfrak{W}(f)$ the number of strong and weak nodal domains of a function f, respectively.

Theorem (Discrete Nodal Domain Theorem). Let $G = (V, E)$ be finite, simple, connected. Then for any eigenfunction f_k w.r.t. the k-th eigenvalue λ_k with multiplicity r, we have $\mathfrak{W}(f_k) \leq k$ and $\mathfrak{S}(f_k) \leq k + r - 1$.