The proof of the weak nodal domain theorem $\mathfrak{W}(f_k) \le k$ is very inspiring. We are going to discuss how to dig out more from it.

Question. What happens if we apply the strategy to strong nodal domains?

Observation. If we apply the strategy for the proof of $\mathfrak{W}(f_k) \le k$ to strong nodal domains, we have the following.

Let $\Omega_1, \Omega_2, \cdots, \Omega_m$ be the strong nodal domains of f_k.

Let $g_i(x) := \begin{cases} f_k(x), & \text{if } x \in \Omega_i \\ 0, & \text{otherwise} \end{cases}$, $i = 1, \cdots, m$.

Let $(a_1, \cdots, a_m) \in \mathbb{R}^m$, $a(x) := \begin{cases} a_i, & \text{if } x \in \Omega_i \\ 0, & \text{otherwise.} \end{cases}$

(FACT 1). If $m \ge k$, then there exists $0 \ne (a_1, \cdots, a_m) \in \mathbb{R}^m$, s.t. $g = a f_k$ (i.e. $g = \sum_{i=1}^{m} a_i g_i$) is an eigenfunction to λ_k, i.e. $\Delta g + \lambda_k g = 0$.

(FACT 2). If Ω_i and Ω_j are adjacent, then $a_i = a_j$.

(FACT 3) If there exists a ~~ver~~ zero vertex which is adjacent to exactly two strong nodal domains Ω_i and Ω_j, then $a_i = a_j$.

All the arguments are built upon the following estimate.

$$\lambda_m \le \frac{(dg, dg)}{(g, g)} \le \lambda_k + \frac{1}{(g,g)} \sum_{\{x,y\} \in E} (a(x) - a(y))^2 f_k(x) f_k(y) \le \lambda_k.$$

A first observation is that if f_k has no zero vertex and G connected, then $\mathfrak{S}(f_k) \le k$. On the one hand, the strong nodal domains coincide with the weak nodal domains in the case. On the other hand, any two strong nodal domains Ω_i, Ω_j can be connected by a "strong nodal domain path" and hence $a_i = a_j \ \forall i, j$.

In fact, we can state more.

Theorem 1. If f_k has $k+s$ strong nodal domains, $s \geq 1$, then $G \setminus \{v : f_k(v) = 0\}$ consists of at least $s+1$ connected components.

[Gladwell, Zhu] Courant's nodal line theorem and its discrete counterparts, Q. Jl Mech. Appl. Math. (2002) 55 (1), 1-15.

Notice that an immediate consequence of Theorem 1. is

Corollary 1. If G is connected and f_k has no zero vertex, then $\mathfrak{S}(f_k) \leq k$.

Proof. Suppose $\mathfrak{S}(f_k) > k$, then $\mathfrak{S}(f_k) \geq k+1$, then G has at least 2 connected components by Thm 1. This contradicts to the connectedness of G. $\square$

Proof of Theorem 1.

By (FACT 1), we can construct a nonzero eigenfunction g to λ_k as $g = \sum_{i=1}^{k+s} a_i g_i$

Moreover, we can set s of $\{a_1, \ldots, a_{k+s}\}$ to be zero.

Suppose that $G \setminus \{v : f_k(v) = 0\}$ consists of $\leq s$ connected components. Then we can choose the s of $\{a_1, \ldots, a_{k+s}\}$ s.t. $\Omega_{i_1}, \ldots, \Omega_{i_s}$ runs over all connected components of $G \setminus \{v : f_k(v) = 0\}$. Now, by (FACT 2), we have each $a_i = 0$. This contradicts to the fact that g is nonzero. $\square$

Theorem 1 has very interesting consequences.

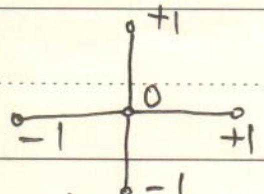
<u>Corollary 2</u>. Each connected component of $G \setminus \{v : f_k(v) = 0\}$ has at most k strong nodal domains.

<u>Proof</u>. Suppose a connected component of $G \setminus \{v : f_k(v) = 0\}$ has $\geq k+1$ strong nodal domains. Thm 1 tells $G \setminus \{v : f_k(v) = 0\}$ consists of at least 2 connected components.
Let us denote the number of connected components of $G \setminus \{v : f_k(v) = 0\}$ by $s+1$, $s \geq 1$. Then f_k has $\geq k+1+s$ strong nodal domains. Thm 1 tells $G \setminus \{v : f_k(v) = 0\}$ has at least $s+2$ connected components. Contradiction. □

Next, we dig out more from (FACT 3).
[Yong Lin, Gábor Lippner, Dan Mangoubi, Shing-Tung Yau, Nodal geometry of graphs on surfaces, arXiv: 1307.3226].

<u>Theorem</u> 2. Let G be connected finite simple graph with maximal vertex degree d_{max}. Then
$$\mathfrak{S}(f_k) \leq d_{max} \cdot (k-1).$$

<u>Remark</u>. Recall the example of a star graph



~~Theo~~ For f_2 given as in the figure, we have $\mathfrak{S}(f_2) = 4 = d_{max} \cdot (2-1)$. Therefore, the estimate is sharp.

Recall (FACT 1) - (FACT 3), we introduce the "Nodal geography" of f_k according to the values of a_i's. First, consider the graph $G^{snd} = (V^{snd}, E^{snd})$, when V^{snd} is the set of strong nodal domains of f_k, E^{snd} is the set of pairs of adjacent strong nodal domains. If G^{snd} is connected, we can use the proof of $W(f_k) \le k$ to show $\mathfrak{S}(f_k) \le k$. (or use Corollary 2 directly).

~~Th~~ Take any connected component of G^{snd}, the union of the corresponding domains is called a (nodal) ~~domain~~ region of f_k. Clearly, any vertex $x \in V$ not in the region but adjacent to the region is a zero vertex. We call a region small if it consist of a single strong nodal domain. Otherwise, we call it large.

Islands. We construct them recursively: At the beginning, every region is an island. At each step, we look for a zero vertex which is adjacent to exactly two different islands, at least one of which has to be a small island (i.e. an island consists of only one strong nodal domain). We unit these two islands (and the zero vertex) into one big island. We call an island large if it consists of more than one strong nodal domains.

Let us denote by $I_1, \dots, I_s$ the final list of islands. Let $\mathcal{S}$ be the set of small islands and $\mathcal{L}$ be the set of large islands.

Lemma 1 If $m \ge k$, then there exists a non zero eigenfunction $g = a f_k$ (i.e. $g = \sum_{i=1}^{m} a_i g_i$) to λ_k. Moreover, a_i's are constant on each island.

This reminds us to consider the s-dimensional subspace W of $\mathbb{R}^m$ consisting of functions that are constant on each island. Let $\varphi_1, \dots, \varphi_s$ be a basis of W.

Lemma 2. $\mathfrak{S}(f_k) \leq k+s-1$.

Proof: Suppose $\mathfrak{S}(f_k) > k+s-1$. Then we can find

$$g = \sum_{i=1}^{m} a_i g_i = a f_k, \text{ where } \underbrace{(a_1, \cdots, a_m)}_{\neq 0} \in \mathbb{R}^m \text{ s.t.}$$

$$\left\{ \begin{array}{ll} (g, f_1) = \cdots = (g, f_{k-1}) = 0 \quad \text{and} \ (*1) & \\ \langle (a_1, \cdots, a_m), \psi_\ell \rangle_{\text{Eucl.}} = 0, \quad \ell = 1, \cdots, s. \ (*2) & \end{array} \right. \quad (*)$$

This is achievable: We can write $(*)$ more explicitly:

$$\left\{ \begin{array}{l} 0 = \sum_{x \in V} g(x) f_j(x) d_x = \sum_{x \in V} \sum_{i=1}^{m} a_i g_i(x) f_j(x) d_x \\ \quad = \sum_{i=1}^{m} a_i \left(\sum_{x \in V} g_i(x) f_j(x) d_x \right), \quad j = 1, \cdots, k-1 \\ 0 = \sum_{i=1}^{m} a_i \psi_\ell(\Omega_i), \quad \ell = 1, \cdots, s \end{array} \right.$$

We have $k+s-1$ equations, and $m > k+s-1$ variables.
Therefore, there must exist a nonzero solution.

By previous (FACT1) - (FACT3), we derive from $(*1)$ that g is an eigenfunction to λ_k, and a_i's are constant on each island. That is $(a_1, \cdots, a_m) \in W \leq \mathbb{R}^m$.
However, $(*2)$ implies that $(a_1, \cdots, a_m)$ is orthogonal to W.
Therefore, $(a_1, \cdots, a_m) = 0$. This is a contradiction. □

~~If we~~ How do $\mathfrak{S}(f_k)$ and s compare? Recall

$$s = |\mathcal{L}| + |\mathcal{S}| \quad (\# \text{ of islands})$$

$$\mathfrak{S}(f_k) \geq 2|\mathcal{L}| + |\mathcal{S}|$$

However, if we simply use $s \leq \mathfrak{S}(f_k)$, we obtain

$$\mathfrak{S}(f_k) \leq k + \mathfrak{S}(f_k) - 1 \text{ which is trivial.}$$

We observe that there is subspaces of W that $(a_1, \cdots, a_m)$ is automatically orthogonal to.

Let $v \in V$ be a zero vertex of f_k. ~~Then we have~~

Let $g = \sum_{i=1}^{m} a_i g = a f_k$ be the eigenfunction to λ_k obtained as above. Then

$$0 = -\lambda_k g(v) = (\Delta g)(v) = \frac{1}{d_v} \sum_{x: x \sim v} (g(x) - g(v)) = \frac{1}{d_v} \sum_{x: x \sim v} g(x)$$

$$= \frac{1}{d_v} \sum_{x: x \sim v} \sum_{i=1}^{m} a_i g_i(x)$$

Let $J_1, \cdots, J_p$ be the islands adjacent to v, then

$$0 = \sum_{x: x \sim v} \sum_{i=1}^{m} a_i g_i(x) = \sum_{j=1}^{p} \sum_{\substack{x: x \sim v \\ x \sim J_j}} a(J_j) f_k(x)$$

$$= \sum_{j=1}^{p} a(J_j) \sum_{\substack{x: x \sim v \\ x \sim J_j}} f_k(x).$$

$$(*) \quad = \sum_{j=1}^{p} \sum_{\Omega \in J_j} a(\Omega) \Bigg(\sum_{\substack{x: x \sim v \\ x \sim J_j}} f_k(x) \Bigg) \cdot \frac{1}{|J_j|},$$ where $|J_j|$ is the number of strong nodal domains in J_j

We introduce a function $\varphi_v \in W$, s.t. for any strong nodal domain Ω,

$$\varphi_v(\Omega) = \frac{1}{|I(\Omega)|} \sum_{\substack{x \in I(\Omega) \\ x \sim v}} f_k(x)$$

where $I(\Omega)$ stands for the island in which Ω lies.

If the set $\{x \in I(\Omega) : x \sim v\} = \phi$, set $\varphi_v(\Omega) = 0$.

We notice that $\varphi_v \in W$.

Moreover $(*)$ can be written as

$$0 = \sum_{i=1}^{m} a(\Omega_i) \varphi_v(\Omega_i)$$

That is, $(a_1, \dots, a_m)$ is orthogonal to the linear subspace $\mathrm{span}\{\varphi_v : v \text{ is a zero vertex of } f_k\} \leq W$.

In order to have a nontrivial linear subspace, we hope $\varphi_v \neq 0$. This can be assured when v is adjacent to at least one small island. Let $\Omega_0 \in S$ be such a small island, then we know

$$\varphi_v(\Omega_0) = \sum_{\substack{x \in \Omega_0 \\ x \sim v}} \underbrace{f_k(x)}_{\text{with the same sign}} \neq 0.$$

Therefore, we like to consider the set

$$V_0 \subset V := \{ x \in V : f_k(x) = 0, \ x \text{ is adjacent to at least one small island.} \}$$

Then the linear subspace

$$W_0 = \mathrm{span}\{ \varphi_v : v \in V_0 \} \leq W.$$

We know $(a_1, \dots, a_m)$ constructed above is automatically orthogonal to W_0.

We hope to know the dimension of W_0.

Lemma3: The dimension of the linear subspace $W_0 \leq W$ is at least $\dfrac{|S|}{d_{max}}$.

Proof: We successively pick vertices $v_1, v_2, \dots \in V_0$ with the property that for every i, the vertex v_i is adjacent to a small island that was not adjacent to any previously picked vertex. Recall that for a small island $\Omega_0 \in S$ that ~~this~~ $v \in V_0$ is adjacent to, we have $\varphi_v(\Omega_0) \neq 0$. Therefore, our process guarantees that all φ_{v_i} are linearly independent.

In each step, we find a small island that is not adjacent to any of the previously selected ~~$v \in V_0$~~ verties in V_0, and choose any adjacent ~~ver~~ zero vertex as the next v_i. Therefore the number of small islands we can choose decrease at most by d_{max} each ~~time~~ step. Hence, the sequence $v_1, v_2, \cdots$ is of length $\geq \frac{|S|}{d_{max}}$. $\square$

Lemma 4. Let l_0 be the codimension of W_0 in W. Then we have $l_0 \leq \frac{d_{max}-1}{d_{max}} \mathfrak{S}(f_k)$.

Proof. Observe first that

$$\mathfrak{S}(f_k) \geq 2|\mathcal{L}| + |\mathcal{S}|.$$

(since each large island contains at least two strong domains)

On the other hand

$$\begin{aligned} l_0 &= |\mathcal{L}| + |\mathcal{S}| - \dim W_0 \\ &\leq |\mathcal{L}| + |\mathcal{S}| - \frac{|\mathcal{S}|}{d_{max}} \\ &\leq 2\frac{d_{max}-1}{d_{max}}|\mathcal{L}| + \frac{d_{max}-1}{d_{max}}|\mathcal{S}| \\ &\leq \frac{d_{max}-1}{d_{max}} \mathfrak{S}(f_k) \end{aligned}$$

$\left(\text{Since } d_{max} \geq 2 \Rightarrow 2\frac{d_{max}-1}{d_{max}} \geq 1 \right)$

If $d_{max} = 1$, G connected $\Rightarrow G = K_2$. $\square$

Proof of Thm 2. We just modify the proof of Lemma 2. Let $\psi_1, \cdots, \psi_{l_0}$ be a basis of the orthogonal complement of W_0 in W.

Suppose $\mathfrak{S}(f_k) > k + l_0 - 1$. Then we can find

$0 \neq (a_1, \cdots, a_m) \in \mathbb{R}^m$, $g = \sum_{i=1}^{m} a_i g_i = a f_k$ s.t.

$$\begin{cases} (g, f_j) = 0, \quad j = 1, \cdots, k-1 & (\maltese 1) \\ \langle (a_1, \cdots, a_m), \psi_l \rangle = 0, \quad l = 1, \cdots, l_0 . & (\maltese 2) \end{cases} \qquad (\maltese)$$

On the other hand, (FACT 1) – (FACT 3) imply that g is an eigenfct to λ_k, and a_i's are constant on each island. Moreover, $(\maltese 2)$ tells that $(a_1, \cdots, a_m)$ ~~is orthogonal to~~ lie in W_0. However, the analysis on page 56 before Lemma 3 tells that $(a_1, \cdots, a_m)$ are orthogonal to W_0. Therefore, $(a_1, \cdots, a_m) = 0$, which is a contradiction.

Hence, we derive

$$\mathfrak{S}(f_k) \leq k + l_0 - 1 .$$

Applying Lemma 4, we have

$$\mathfrak{S}(f_k) \leq k - 1 + \frac{d_{max} - 1}{d_{max}} \mathfrak{S}(f_k)$$

That is, $\mathfrak{S}(f_k) \leq d_{max}(k-1)$. □

<u>Further reading</u>: Using the same scheme of the proof of Thm 2, Lin-Lippner-Mangoubi-Yau show further that

<u>Theorem 3</u>. Let G be a finite simple 3-connected graph. with genus g, then $\mathfrak{S}(f_k) \leq 6(k-1) + 14(2g-2)$.

. A graph G on more than 2 vertices is said to be k-connected. if there does not exist $k-1$ vertices whose removal disconnects