$0 \neq (a_1, \cdots, a_m) \in \mathbb{R}^m$, $g = \sum_{i=1}^m a_i g_i = a f_k$ s.t.

$$\left\{ \begin{array}{ll} (g, f_j) = 0, \quad j = 1, \cdots, k-1 & (\maltese 1) \\ & \quad (\maltese) \\ \langle (a_1, \cdots, a_m), \psi_l \rangle = 0, \quad l = 1, \cdots, l_0. & (\maltese 2) \end{array} \right.$$

On the other hand, (FACT 1) – (FACT 3) imply that g is an eigenfct to λ_k, and a_i's are constant on each island. Moreover, ($\maltese$2) tells that $(a_1, \cdots, a_m)$ ~~is orthogonal to~~ lie in W_0. However, the analysis on page 56 before Lemma 3 tells that $(a_1, \cdots, a_m)$ are orthogonal to W_0. Therefore, $(a_1, \cdots, a_m) = 0$, which is a contradiction. Hence, we derive

$$\mathfrak{S}(f_k) \leq k + l_0 - 1 \quad \left(\text{i.e. } \mathfrak{S}(f_k) \leq k - 1 + (s - \dim W_0) \right)$$

Applying Lemma 4, we have

$$\mathfrak{S}(f_k) \leq k - 1 + \frac{d_{max} - 1}{d_{max}} \mathfrak{S}(f_k)$$

That is, $\mathfrak{S}(f_k) \leq d_{max}(k-1)$. $\square$

Further reading: Using the same scheme of the proof of Thm 2, Lin-Lippner-Mangoubi-Yau show further that

Theorem 3. Let G be a finite simple 3-connected graph. with genus g, then $\mathfrak{S}(f_k) \leq 6(k-1) + 14(2g-2)$.

· A graph G on more than 2 vertices is said to be k-connected if there does not exist $k-1$ vertices whose removal disconnects the graph.

- Genus of a graph.

[J.W.T. Youngs] Minimal imbeddings and the genus of a graph. Journal of Mathematics and Mechanics, 1963 vol 12, no. 2, 303-315.

Any connected graph can be imbedded in an orientable 2-manifold. For example, one can select $|V|$ distinct points on a 2-sphere and then accomodate each edge on a "handle" attached appropriately to the 2-sphere. An imbedding is called minimal if there is no other imbedding in an orientable 2-mfld of lower genus. The genus of the graph is defined to be the genus of the orientable 2-manifold in which the graph is minimally imbedded. Youngs showed that if a connected graph G is minimally imbedded in a 2-mfld, then the imbedding is a 2-cell imbedding. Therefore $|V| - |E| + |F| = 2-2g$ holds.

Lin-Lippner-Mangoubi-Yau show that for 3-connected graph G with genus g, $\dim W_0 \geq \frac{1}{6}(181 - 14(2g-2))$.

Then, similarly as in the proof of Lemma 4, they derive $\operatorname{codim} W_0 \leq \frac{5}{6}\mathfrak{S}(f_n) + \frac{14}{6}(2g-2)$.

Then the scheme of Thm 2's proof implies Thm 3. $\square$

~~Lemma 5~~. Let us restructure a bit our proofs.

Recall $V_0 := \{ x \in V : f_k(x) = 0,\ x \text{ is adjacent to at least one small island} \}$

$\forall v \in V_0$, Define

$$\varphi_v(\Omega) = \begin{cases} \dfrac{1}{|I(\Omega)|} \displaystyle\sum_{\substack{y : y \sim v \\ y \in I(\Omega)}} f_k(y), & \text{if } I(\Omega) \text{ is adjacent to } v \\ 0, & \text{otherwise.} \end{cases}$$

$$W_0 = \operatorname{span}\{ \varphi_v : v \in V_0 \}$$

Lemma 5 $\mathfrak{S}(f_k) \le k - 1 + \underbrace{s}_{\text{\# of islands.}} - \dim W_0$.

Proof. See pp. 58.

Lemma 6. For 3-connected graph G with genus g, we have $\dim W_0 \ge \frac{1}{6}\left(|\mathcal{S}| - 14(2g-2) \right)$

Lemma 7. For 3-connected graph G with genus g, we have $\operatorname{codim} W_0 \le \frac{5}{6} \mathfrak{S}(f_k) + \frac{14}{6}(2g-2)$.

Proof of Lemma 7.

$$\begin{aligned} \operatorname{codim} W_0 = s - \dim W_0 &= |\mathcal{L}| + |\mathcal{S}| - \tfrac{1}{6}|\mathcal{S}| + \tfrac{14}{6}(2g-2) \\ &\le 2 \cdot \tfrac{5}{6}|\mathcal{L}| + \tfrac{5}{6}|\mathcal{S}| + \tfrac{14}{6}(2g-2) \\ &\le \tfrac{5}{6}\mathfrak{S}(f_k) + \tfrac{14}{6}(2g-2) \qquad \square \end{aligned}$$

Combining Lemma 5 and Lemma 7, we arrive at

$$\mathfrak{S}(f_k) \le k-1 + \tfrac{5}{6}\mathfrak{S}(f_k) + \tfrac{14}{6}(2g-2)$$

That is, $\frac{1}{6}\mathfrak{S}(f_k) \le (k-1) + \frac{14}{6}(2g-2)$. This proves Thm 3.

Next, we explain the proof of Lemma 6.

Proof of Lemma 6.

Let g be the genus of G. Consider ~~a~~ a new bipartite graph H as follows: Denote $\mathcal{S} = \{i_1, \dots, i_{|\mathcal{S}|}\}$ the set of small islands. The vertex set of H is given by $\mathcal{S} \cup V_0$. The edges set is $\{\{i_j, v\} : v \in V_0, j = 1, \dots, |\mathcal{S}|, v \text{ is adjacent to } i_j \text{ in } G\}$.

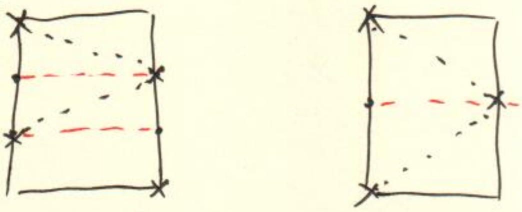
Observe that H can be obtained from G by deleting vertices, and edges, and contracting a connected component to a vertex. Hence the genus g_H of H satisfies $g_H \le g$.

G is 3-connected implies that for any i_j, there exists at least 3 vertices in V_0 adjacent to it. (Observe that for small islands, ~~the only~~ all ~~paas~~ vertices adjacent to them are lying in V_0). That means, each i_j has vertex degree ≥ 3 in H.

Claim: If $|\mathcal{S}| > 14(2g_H - 2)$, then $\exists\, v \in V_0$ whose degree in H is at most 6.

Proof of the claim: ~~The~~ Take a minimal imbedding of H in Σ_{g_H}. By Youngs' result, it is a 2-cell imbedding, that is, every connected component of $\Sigma_{g_H} \setminus H$ is a disc. Since
i.e., every face

H is bipartite; the bdy of each face is a cycle of even length at least 4. Observe, we can always cut a face into smaller faces of length = 4 by adding edges and keep it bipartite.



Note that, this only increase some vertex degrees.
We can transform the graph H to a graph H_1.
The vertex set is $\mathcal{V}$. Observe that in every face of H, there are two vertices from $\mathcal{V}$ on a diagonal position.
The edge set is the lines connecting such two vertices in each face. Recall the vertex degree of i_j is at least 3 in H, and therefore at least 3 faces adjacent to it, and furthermore, the vertex degree of i_j in H_1 is still ≥ 3.
In Σ_{g_H}, the faces of H_1 corresponds exactly to vertices in V_0.

Suppose every degree of a V_0 is ≥ 7, then each face of H_1 has ≥ 7 sides. Hence we have

$$e \geq \frac{7f}{2}, \quad e \geq \frac{3v}{2}.$$

Recall $v - e + f = 2 - 2g$, we have

$$e = v + f + 2g - 2$$

$$e \geq a \cdot \frac{3v}{2} + (1-a)\frac{7f}{2}, \quad \text{for some } a \in (0,1)$$

$$\Rightarrow \left(\frac{3a}{2} - 1\right) v \leq \left(1 - \frac{7(1-a)}{2}\right) f + 2g - 2.$$

$$1 - \frac{7(1-a)}{2} = 0 \Rightarrow a = \frac{5}{7} \Rightarrow \frac{3a}{2} - 1 = \frac{1}{14}$$

$\Rightarrow \underset{|\delta|}{\underset{\parallel}{v}} \leq 14(2g-2)$. This is a contradiction. $\square$

Now we come back to the proof of $\dim W_0 \geq \frac{1}{6}(|\delta| - 14(2g-2))$

If $|\delta| \leq 14(2g-2)$, there is nothing to prove.

If $\underset{\geq 14(2g_H - 2)}{|\delta| > 14(2g-2)}$, by the claim, $\exists\, v \in V_0$ ~~whose~~ which is adjacent to at most 6 small islands. set $v = v_1$, and remove v_1 and all its adjacent small islands in H. This cannot increase the genus of the graph H. We repeat the process until the size of $|\delta|$ shrink below $14(2g-2)$. In each step, we lose at most 6 from δ. Hence we can find the sequence $v_1, v_2, \dots$ of length $\geq \frac{|\delta| - 14(2g-2)}{6}$.

Therefore, $\dim W_0 \geq \frac{1}{6}(|\delta| - 14(2g-2))$. $\square$.