Discussions about Remark 1 on page 71. Solutions of discrete-time heat eq. and distributions of a random walker.

Let $f_0 = \delta_{x_0}$, and $f : V \times \{0,1,2,\cdots\} \to \mathbb{R}$ satisfies

$$\begin{cases} f(x,t+1) - f(x,t) = \Delta f(x,t), & t = 0,1,2,\cdots, x \in V \\ f(x,0) = f_0(x) \end{cases}$$

Recall the solution $f(\cdot,t) = P^t f_0(\cdot)$.

We calculate for $y \in V$,

$$f(y,t) = P^t f_0(y) = P(P^{t-1} f_0)(y)$$

$$= \frac{1}{d_y} \sum_{\substack{y_1 \in V \\ y_1 \sim y}} (P^{t-1} f_0)(y_1) = \frac{1}{d_y} \sum_{\substack{y_1 \in V \\ y_1 \sim y}} P(P^{t-2} f_0)(y_1)$$

$$= \frac{1}{d_y} \sum_{\substack{y_1 \in V \\ y_1 \sim y}} \frac{1}{d_{y_1}} \sum_{\substack{y_2 \in V \\ y_2 \sim y_1}} (P^{t-2} f_0)(y_2)$$

$$= \cdots$$

$$= \sum_{\substack{y_1 \in V \\ y_1 \sim y}} \sum_{\substack{y_2 \in V \\ y_2 \sim y_1}} \cdots \sum_{\substack{y_{t-1} \in V \\ y_{t-1} \sim y_{t-2}}} \sum_{\substack{y_t \in V \\ y_t \sim y_{t-1}}} \frac{1}{d_y} \cdot \frac{1}{d_{y_1}} \cdots \frac{1}{d_{y_{t-2}}} \cdot \frac{1}{d_{y_{t-1}}} f_0(y_t)$$

Notice that $f_0(y_t)$ is only nonzero at $y_t = x_0$. Therefore

$$f(y,t) = \sum_{\substack{y_1, \cdots, y_{t-1} \in V \\ y \sim y_1 \sim y_2 \cdots \sim y_{t-2} \sim y_{t-1} \sim x_0}} \frac{1}{d_y} \frac{1}{d_{y_1}} \cdots \frac{1}{d_{y_{t-1}}}$$

$= \mathbb{P}$ (The random walker arrives at $x_0 \in V$ after t steps starting from $y \in V$) $=: \mathbb{P}_y(X_t = x_0)$

$$= \frac{d_{x_0}}{d_y} \sum_{\substack{y_1, \cdots, y_{t-1} \in V \\ y \sim y_1 \sim \cdots \sim y_{t-1} \sim x_0}} \frac{1}{d_{y_1}} \cdots \frac{1}{d_{y_{t-1}}} \frac{1}{d_{x_0}}$$

$= \frac{d_{x_0}}{d_y} \cdot \mathbb{P}$(The random walker arrives at $y \in V$ after t steps starting from $x_0 \in V$)

$=: \frac{d_{x_0}}{d_y} \mathbb{P}_{x_0}(X_t = y)$

(In fact, the above fact can be written as

$$d_y \, \mathbb{P}_y(X_t = x_0) = d_{x_0} \, \mathbb{P}_{x_0}(X_t = y)$$

This is certain symmerty property. In the term of Markov chains, this is called the reversibility.)

Therefore, Theorem 1 implies

$$\left\| \frac{d_{x_0}}{d_\cdot} \mathbb{P}_{x_0}(X_t = \cdot) - \frac{d_{x_0}}{vol(G)} \right\| \le \rho^n \cdot \sqrt{d_{x_0}}$$

$$\Rightarrow \sum_{y \in V} \left(\frac{d_{x_0}}{d_y} \mathbb{P}_{x_0}(X_t = y) - \frac{d_{x_0}}{vol(G)} \right)^2 d_y \le \left(\rho^n \sqrt{d_{x_0}} \right)^2$$

$$\Rightarrow \sum_{y \in V} \left(\mathbb{P}_{x_0}(X_t = y) - \frac{d_y}{vol(G)} \right)^2 \frac{d_{x_0}^2}{d_y} \le \left(\rho^n \sqrt{d_{x_0}} \right)^2$$

<u>Corollary</u>. Let $G = (V, E)$ be a finite simple connected graph, Let $\{X_t\}$ be the associated simple random walk. Then we have

$$\sum_{y \in V} \left(\mathbb{P}_{x_0}(X_t = y) - \frac{d_y}{vol(G)} \right)^2 \frac{d_{x_0}}{d_y} \le \rho^{2n}.$$

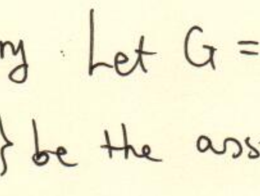
In particular, we have $\left| \mathbb{P}_{x_0}(X_t = y) - \frac{d_y}{vol(G)} \right| \le \rho^n \sqrt{\frac{d_y}{d_{x_0}}}$.

If G is non-bipartite, we have

$$\mathbb{P}_{x_0}(X_t = y) \longrightarrow \frac{d_y}{vol(G)} \quad \text{as } t \to \infty.$$

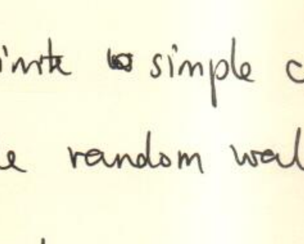
<u>Example 1</u>.

C_5.



<u>limit</u>: $\frac{1}{5}$

K_5



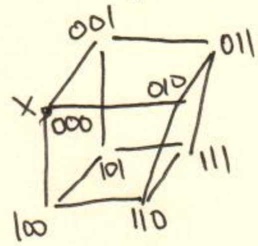
<u>limit</u>: $\frac{4}{20} = \frac{1}{5}$

	0	1	2	3	4
t=0	0	0	1	0	0
t=1	0	1/2	0	1/2	0
t=2	1/4	0	1/2	0	1/4
t=3	1/8	3/8	0	3/8	1/8
t=4	1/4	1/16	3/8	1/16	1/4

	0	1	2	3	4
t=0	0	0	1	0	0
t=1	1/4	1/4	0	1/4	1/4
t=2	3/16	3/16	1/4	3/16	3/16
t=3	13/64	13/64	3/16	13/64	13/64
t=4	51/256	51/256	13/64	51/256	51/256

The convergence is much faster in K_5 than in C_5.

Example 2. 3-Hypercube.



	x_0 = 000	110	101	011	100	010	001	111
$t=0$	1	0	0	0	0	0	0	0
$t=1$	0	0	0	0	$1/3$	$1/3$	$1/3$	0
$t=2$	$1/3$	$2/9$	$2/9$	$2/9$	0	0	0	0
$t=3$	0	0	0	0	$7/27$	$7/27$	$7/27$	$2/9$
$t=4$	$7/27$	$20/81$	$20/81$	$20/81$	0	0	0	0

We see $\mathbb{P}_{x_0}(X_t = \cdot)$ does not converge.

Theorem 3. Let $G=(V,E)$ be a bipartite graph. Let V^+, V^- be a bipartition of V. Then for any function f_0 on V, consider the function $\tilde{f}_0$ on V that takes two values as follows:

$$\tilde{f}_0(x) = \begin{cases} \dfrac{2}{vol(G)} \displaystyle\sum_{y \in V^+} f_0(y)\,dy, & \text{if } x \in V^+. \\[2ex] \dfrac{2}{vol(G)} \displaystyle\sum_{y \in V^-} f_0(y)\,dy, & \text{if } x \in V^-. \end{cases}$$

Then, we have for all even t

$$\|f(\cdot, t) - \tilde{f}_0\| \leq \rho^n \|f_0\|$$

where $\rho = \max(|1-\lambda_2|, |\lambda_{N-1} - 1|) = 1-\lambda_2$.

Consider the function $\tilde{\tilde{f}}_0$ on V that takes two values as follows

$$\tilde{\tilde{f}}_0(x) = \begin{cases} \dfrac{2}{vol(G)} \displaystyle\sum_{y \in V^-} f_0(y)\,dy, & \text{if } x \in V^+ \\[2ex] \dfrac{2}{vol(G)} \displaystyle\sum_{y \in V^+} f_0(y)\,dy, & \text{if } x \in V^-. \end{cases}$$

Then we have for all odd t

$$\|f(\cdot, t) - \tilde{\tilde{f}}_0\| \leq \rho^n \|f_0\|$$

Proof: This follows from a similar strategy as in the proof of Thm 1. Let $\{f_k\}_{k=1}^N$ be an orthogonal basis of eigenfunctions. Then

$$P^t f_0 = \sum_{k=1}^N a_k (1-\lambda_k)^t f_k, \quad \text{where } a_k = (f, f_k)$$

Recall $a_1 f_1 = \frac{1}{vol(G)} \sum_{y \in V} f_0(y)\, dy$.

Observe that

$$f_N(x) = \begin{cases} \frac{1}{\sqrt{vol(G)}}, & \text{if } x \in V_0^+ \\ -\frac{1}{\sqrt{vol(G)}}, & \text{if } x \in V_0^- \end{cases}$$

is the eigenfunction to $\lambda_N = 2$.

Therefore

$$a_N = (f_0, f_N) = \sum_{y \in V} f_0(y)\, f_N(y)\, dy$$

$$= \frac{1}{\sqrt{vol(G)}} \left(\sum_{y \in V^+} f_0(y)\, dy - \sum_{y \in V^-} f_0(y)\, dy \right).$$

Hence

$$a_N f_N(x) = \begin{cases} \frac{1}{vol(G)} \left(\sum_{y \in V^+} f_0(y)\, dy - \sum_{y \in V^-} f_0(y)\, dy \right) \\ \frac{1}{vol(G)} \left(\sum_{y \in V^-} f_0(y)\, dy - \sum_{y \in V^+} f_0(y)\, dy \right) \end{cases}$$

Notice that

$$a_1 f_1 + a_N f_N = \tilde{f}_0 \quad \text{and} \quad a_1 f_1 - a_N f_N = \tilde{\tilde{f}}_0 .$$

We then have

$$p^t f_0 = \sum_{k=1}^N (1-\lambda_k)^t a_k f_k = \sum_{k=2}^{N-1} (1-\lambda_k)^t a_k f_k + a_1 f_1 + (1-\lambda_N)^t a_N f_N .$$

Since $(1-\lambda_N)^t = (-1)^t = \begin{cases} 1, & t \text{ even} \\ -1, & t \text{ odd} \end{cases}$, we obtain

$$p^t f_0 = \sum_{k=2}^{N-1} (1-\lambda_k)^t a_k f_k + a_1 f_1 + (-1)^t a_N f_N .$$

if t even, we have

$$\|P^t f_0 - \tilde{f}_0\|^2 = \sum_{k=2}^{N-1} (1-\lambda_k)^{2t} a_k^2 \leq \max_{2\leq k\leq N-1} |1-\lambda_k|^{2t} \cdot \underbrace{\sum_{k=2}^{N-1} a_k^2}_{\leq \|f_0\|^2}$$

whereas t odd,

$$\|P^t f_0 - \tilde{\tilde{f}}_0\|^2 = \sum_{k=2}^{N-1} (1-\lambda_k)^{2t} a_k^2 \leq \max_{2\leq k\leq N-1} |1-\lambda_k|^{2t} \sum_{k=2}^{N-1} a_k^2.$$

This finishes the proof. $\square$

<u>Corollary</u>: Let $G=(V,E)$ be bipartite with $V=V^+\sqcup V^-$.
Let $x_0 \in V^+$. Then for $y\in V$

$$\mathbb{P}_{x_0}(X_t=y) \longrightarrow \begin{cases} \frac{2d_y}{vol(G)}, & \text{if } y\in V^+ \\ 0 & \text{if } y\in V^- \end{cases}$$

as $t\to\infty$, t even

and
$$\mathbb{P}_{x_0}(X_t=y) \longrightarrow \begin{cases} 0, & \text{if } y\in V^+ \\ \frac{2d_y}{vol(G)}, & \text{if } y\in V^- \end{cases}$$

as $t\to\infty$, t odd.

Back to the proof of Cheeger inequality: Observe that, we actually show that there exists $\Omega_1 \subseteq supp(f_+)$, $\Omega_2\subseteq supp(f_-)$, where $\Delta f + \lambda_2 f = 0$, s.t.

$$\max_{i=1,2} \phi(\Omega_i) \leq \sqrt{2\lambda_2}$$

Therefore $\lambda_2 \geq \frac{1}{2}\Big(\min_{\substack{\Omega_1,\Omega_2\subseteq V\\ \Omega_1\cap\Omega_2=\emptyset\\ \Omega_i\neq\emptyset}} \max_{i=1,2} \phi(\Omega_i)\Big)^2$.

~~We call Ω_1, Ω_2 a subpartition of V if $\Omega_1,\Omega_2\subseteq V$, $\Omega_1\cap\Omega_2=\emptyset$.~~

<u>Claim</u>: $h = \min_{\substack{\Omega_1,\Omega_2\subseteq V\\ \Omega_1\sqcup\Omega_2=V\\ \Omega_1,\Omega_2\neq\emptyset}} \max_{i=1,2}\phi(\Omega_i) = \min_{\substack{\Omega_1,\Omega_2\subseteq V\\ \Omega_1\cap\Omega_2=\emptyset\\ \Omega_1,\Omega_2\neq\emptyset}} \max_{i=1,2}\phi(\Omega_i)$.

Proof: It is enough to show any ~~subpartition~~ Ω_1, Ω_2 achieving the $\min\limits_{\substack{\Omega_1,\Omega_2 \subseteq V \\ \Omega_1\cap\Omega_2=\phi \\ \Omega_1,\Omega_2 \neq \phi}} \max\limits_{i=1,2} \phi(\Omega_i)$ can be replaced by a partition: i.e., $\Omega_1', \Omega_2' \subseteq V$, $\Omega_1' \sqcup \Omega_2' = V$. W.l.o.g., suppose $\phi(\Omega_1) \geq \phi(\Omega_2)$.

Case (i): if $vol(\Omega_1) \leq \frac{1}{2} vol(G)$, then $\phi(\Omega_1) \geq \phi(\Omega_1^c)$

Hence $\max\{\phi(\Omega_1), \phi(\Omega_2)\} = \max\{\phi(\Omega_1), \phi(\Omega_1^c)\} = \phi(\Omega_1)$.

Case (ii) if $vol(\Omega_1) > \frac{1}{2} vol(G)$, then $vol(\Omega_2) \leq \frac{1}{2} vol(G)$.

Hence $\phi(\Omega_2) \geq \phi(\Omega_2^c)$.

Therefore, $\max\{\phi(\Omega_2), \phi(\Omega_2^c)\} = \phi(\Omega_2) \leq \max\{\phi(\Omega_1), \phi(\Omega_2)\}$.

This finishes the proof. □

(II.3). λ_N, dual Cheeger constant and bipartiteness

Now, we switch to the other fact that

$$\lambda_N = 2 \quad \text{iff} \quad G \text{ is bipartite.}$$

Like in the case of (II.2), we hope to quantify how close a graph is to be bipartite.

For any $\Omega \subseteq V$, we denote

$$E(\Omega) := E(\Omega,\Omega) = \sum_{u\in\Omega} \sum_{\substack{v\in\Omega \\ v\sim u}} 1 = 2 \cdot \#\text{ edges with both adj. vertices lying in } \Omega.$$

Definition (dual Cheeger constant / bipartiteness ratio)

Given a finite simple graph $G=(V,E)$, its bipartiteness ratio is defined by

$$\beta = \min_{(V_1,V_2)} \frac{E(V_1) + E(V_2) + |\partial(V_1 \cup V_2)|}{vol(V_1 \cup V_2)}$$

where the minimum is taken over all possible $V_1, V_2 \subseteq V$ s.t. $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 \neq \phi$.

Remark: Notice particularly that we allow one of V_1 and V_2 to be empty. Moreover, it is not required that $V_1 \cup V_2 = V$.

Theorem 4 (dual Cheeger inequality). We have

$$\frac{\beta^2}{2} \leq 2 - \lambda_N \leq 2\beta.$$

Lemma II.3.1. We have

$$2 - \lambda_N = \min_{f \neq 0} \frac{\sum_{\{u,v\}} (f(u)+f(v))^2}{\sum_u f(u)^2 d_u}.$$

Proof: Recall

$$\lambda_N = \max_{f \neq 0} R(f) = \max_{f \neq 0} \frac{\sum_{\{u,v\}} (f(u)-f(v))^2}{\sum_u f(u)^2 d_u}$$

Therefore, we have

$$2 - \lambda_N = \min_{f \neq 0} \left(2 - \frac{\sum_{\{u,v\}} (f(u)-f(v))^2}{\sum_u f(u)^2 d_u} \right)$$

$$= \min_{f \neq 0} \frac{2 \sum_u f(u)^2 d_u - \sum_{\{u,v\}} (f(u)-f(v))^2}{\sum_u f(u)^2 d_u}.$$

Observe that

$$\sum_{\{u,v\}} (f(u)-f(v))^2 = \frac{1}{2} \sum_u \sum_{v: v \sim u} \left(f(u)^2 - 2f(u)f(v) + f(v)^2 \right)$$

$$= \frac{1}{2} \left[\sum_u f(u)^2 d_u \right] - \frac{1}{2} \sum_u \sum_{v: v \sim u} 2f(u)f(v) + \frac{1}{2} \sum_v f(v)^2 d_v$$

$$= \sum_u f(u)^2 d_u - \frac{1}{2} \sum_u \sum_{v: v \sim u} 2f(u)f(v).$$