HOMEWORK 1: RIEMANNIAN METRICS AND ENERGY FUNCTIONAL

RIEMANNIAN GEOMETRY, SPRING 2021

1. (Spheres)

The **sphere**

$$S^n := \left\{ (x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}$$

is a manifold with the following atlas $\{U_\alpha, y_\alpha\}_{\alpha \in \{1,2\}}$:

$$y_1 : \ U_1 := S^n \setminus \{(0, \dots, 0, 1)\} \longrightarrow \mathbb{R}^n,$$

$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y_1^1, \dots, y_1^n) := \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}} \right).$$

and

$$y_2 : \ U_2 := S^n \setminus \{(0, \dots, 0, -1)\} \longrightarrow \mathbb{R}^n,$$

$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y_2^1, \dots, y_2^n) := \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}} \right).$$

(i) Prove that the above atlas $\{(U_\alpha, y_\alpha)\}_{\alpha \in \{1,2\}}$ is differentiable.

(ii) Let g be the induced metric of S^n from the standard Euclidean metric of $\mathbb{R}^{n+1}$. Prove that in each chart (U_α, y_α), the metric matrix $\left(g_{ij}^{y_\alpha}\right)$ is given by

$$g_{ij}^{y_\alpha} = \frac{4}{(1 + \sum_{i=1}^n (y_\alpha^i)^2)^2} \delta_{ij}.$$

2. (Hyperbolic spaces)

The **hyperboloid** is

$$H^n := \left\{ (x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} (x^i)^2 - (x^{n+1})^2 = -1, x^{n+1} > 0 \right\}.$$

Consider the following map

$$y : \ H^n \longrightarrow B_1(0) := \left\{ (y^1, \dots, y^n) \in \mathbb{R}^n : \sum_{i=1}^{n} (y^i)^2 < 1 \right\} \subset \mathbb{R}^n,$$

$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y^1, \dots, y^n) := \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}} \right).$$

(i) Prove that the above map y is a diffeomorphism between H^n and $B_1(0)$. Therefore, $\{(H^n, y)\}$ is a differentiable atlas of H^n.

(ii) Let g be the Riemannian metric of H^n induced from $\mathbb{R}^{n+1}$ assigned with the *Lorentz metric*:

$$g_L = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n - dx^{n+1} \otimes dx^{n+1}.$$

Prove that in the global chart $\{H^n, y\}$, the metric matrix (g_{ij}) is given by

$$g_{ij} = \frac{4}{(1-\sum_{i=1}^n (y^i)^2)^2}\delta_{ij}.$$

Remark: *Lorentz manifolds* are the spaces occurring in general relativity. For example, the above mentioned $(\mathbb{R}^{n+1}, g_L)$ is a special Lorentz manifold, which is often referred to as a *Minkowski space.* A tangent vector v of a Lorentz manifold can have *negative, positive, or vanishing* norm $\|v\| := \sqrt{g_L(v,v)}$, which is called a *time-like, space-like, or light-like* tangent vector, respectively. Submanifolds of Lorentz manifolds whose tangent vectors are all space-like are Riemannian manifolds with respect to the induced metric. The hyperboloid H^n assigned with the induced metric g, which is often referred to as a *hyperbolic space*, is such an example.

3. (Critical point of energy functional)
Let $\Omega \subset \mathbb{R}^m$ be an open domain with the Riemannian metric given by the matrix $(g_{ij}(x))$, $x \in \Omega$. Let $U \subset \mathbb{R}^n$ be an open domain with the Riemannian metric given by the matrix $(h_{\alpha\beta}(y))$, $y \in U$. Let $f : \Omega \to U$ be a smooth map which is a critical point of the following energy functional:

$$E(f) := \frac{1}{2}\int_\Omega g^{ij}(x)h_{\alpha\beta}(f(x))\frac{\partial f^\alpha(x)}{\partial x^i}\frac{\partial f^\beta(x)}{\partial x^j}\sqrt{g(x)}dx,$$

where we use the notation that

$$f(x) = f(x^1, \dots, x^m) = (f^1(x^1, \dots, x^m), \dots, f^n(x^1, \dots, x^m)),$$

and $g(x) = \det(g_{ij}(x))$, dx being the Lebesgue measure. Such a map f is called a *harmonic map* from (Ω, g) to (U, h).

(i) Compute the differential equations that a harmonic map $f : \Omega \to U$ has to satisfy.

(ii) Show that a geodesic in (U, h) is a harmonic map.

(ii) If $U = \mathbb{R}$ with the Euclidean metric, then a harmonic map $f : \Omega \to U$ is called a *harmonic function* on (Ω, g). Write down the differential equations that harmonic functions satisfy.

Remark: Harmonic maps between Riemannian manifolds are canonical objects from the points of view of topology and calculus of variations. These maps provide a rich display of both differential geometric and analytic phenomena. Much of the study of these maps serves as a model for many other challenging problems in geometric analysis and has been athe source of inspiration and undiminishing fascination. ... Harmonic maps with two dimensional domains present special features that are crucial for applications to minimal surfaces (i.e., conformal harmonic maps) and to the deformation theory of Riemann surfaces-Teichmüller theory.... [Preface of "Harmonic maps and their heat flows" by Fanghua Lin and Changyou Wang.]

Harmonic maps into spheres or complex projective spaces have also acquired some physical interest since they turned out to be solutions of the nonlinear $O(N)$ σ-models. For more details, we refer to Misner, Harmonic Maps as Models for Physical Theories, Phys. Rev. D 18 (12) (1978). [Section 1.5 of "Harmonic maps between surfaces" by Jürgen Jost.]