HOMEWORK 4: CURVATURE

RIEMANNIAN GEOMETRY, SPRING 2021

1. (Riemmannian curvature tensor)
Prove that for any $X, Y, Z, W \in \Gamma(TM)$, we have

$$-6\langle R(X,Y)Z,W\rangle = \frac{\partial^2}{\partial s \partial t}_{|s=t=0} [\langle R(X+sZ, Y+tW)(Y+tW), X+sZ\rangle - \langle R(X+sW, Y+tZ)(Y+tZ), X+sW\rangle]$$

(Hint: using the first Bianchi identity.)

Remark: This gives an alternative proof of the following fact: The values

$$\langle R(X,Y)Z,W\rangle, \quad \forall X, Y, Z, W \in \Gamma(TM)$$

are determined by the values

$$\langle R(X,Y)Y,X\rangle, \quad \forall X, Y \in \Gamma(TM).$$

2. (Spheres)
Recall that the **sphere**

$$S^n := \left\{(x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1\right\}$$

is a differentiable manifold with the following differentiable atlas $\{U_\alpha, y_\alpha\}_{\alpha \in \{1,2\}}$:

$$y_1 : \ U_1 := S^n \setminus \{(0, \ldots, 0, 1)\} \longrightarrow \mathbb{R}^n,$$
$$(x^1, \ldots, x^n, x^{n+1}) \mapsto (y_1^1, \ldots, y_1^n) := \left(\frac{x^1}{1-x^{n+1}}, \ldots, \frac{x^n}{1-x^{n+1}}\right).$$

and

$$y_2 : \ U_2 := S^n \setminus \{(0, \ldots, 0, -1)\} \longrightarrow \mathbb{R}^n,$$
$$(x^1, \ldots, x^n, x^{n+1}) \mapsto (y_2^1, \ldots, y_2^n) := \left(\frac{x^1}{1+x^{n+1}}, \ldots, \frac{x^n}{1+x^{n+1}}\right).$$

Recall that the induced metric g of S^n from the standard Euclidean metric of $\mathbb{R}^{n+1}$ is given in local coordinates by

$$g_{ij}^{y_\alpha} = \frac{4}{(1+\sum_{i=1}^n (y_\alpha^i)^2)^2} \delta_{ij}.$$

(i) Compute the sectional curvature, Ricci curvature, and scalar curvature of the sphere S^n. (Recall we have computed the Christoffel symbols in Excercise 2.)

3. (Hyperbolic spaces)
Recall that the **hyperboloid**

$$H^n := \left\{(x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} (x^i)^2 - (x^{n+1})^2 = -1, x^{n+1} > 0\right\}$$

is a differentiable manifold with the following chart:

$$y: \ H^n \longrightarrow B_1(0) := \left\{(y^1, \dots, y^n) \in \mathbb{R}^n : \sum_{i=1}^n (y^i)^2 < 1\right\} \subset \mathbb{R}^n,$$

$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y^1, \dots, y^n) := \left(\frac{x^1}{1+x^{n+1}}, \dots, \frac{x^n}{1+x^{n+1}}\right).$$

Let g be the Riemannian metric of H^n given by

$$g_{ij} = \frac{4}{(1-\sum_{i=1}^n (y^i)^2)^2}\delta_{ij}.$$

(i) Compute the sectional curvature, Ricci curvature, and scalar curvature of the hyperbolic space H^n. (Recall we have computed the Christoffel symbols in Excercise 2.)

4. (The Second Variation Formula for length)

Let $\gamma : [a,b] \to M$ be a smooth curve and

$$F : [a,b] \times (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$$

be a 2-parameter variation of γ. Denote by

$$V(t) := \frac{\partial F}{\partial v}(t,0,0), \quad W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

the two corresponding variational fields. Let $L(v,w) := L(\gamma_{v,w})$ be the length of the curve $\gamma_{v,w}(t) := F(t,v,w), t \in [a,b]$.

(1) Show that

$$\begin{aligned}\frac{\partial^2}{\partial w \partial v} L(v,w) = \int_a^b \frac{1}{\left\|\frac{\partial F}{\partial t}\right\|} \Big\{ &\left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w} \right\rangle - \left\langle R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle \\ &+ \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \\ &- \frac{1}{\left\|\frac{\partial F}{\partial t}\right\|^2} \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle \Big\} dt,\end{aligned}$$

where $\left\|\frac{\partial F}{\partial t}\right\| := \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle^{\frac{1}{2}}$.

(2) Let γ be a normal geodesic. Show that

$$\begin{aligned}\frac{\partial^2}{\partial w \partial v}_{|v=w=0} L(v,w) = &\int_a^b \left(\langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle - T\langle V,T\rangle T\langle W,T\rangle\right) dt \\ &+ \langle \nabla_W V, T\rangle|_a^b,\end{aligned}$$

where $T(t) := \dot\gamma(t)$ is the velocity field along γ.

(3) Consider the orthogonal component $V^\perp, W^\perp$ of V, W with respect to T, that is

$$\begin{aligned}V^\perp &:= V - \langle V,T\rangle T,\\ W^\perp &:= W - \langle W,T\rangle T.\end{aligned}$$

Show that

$$\begin{aligned}\frac{\partial^2}{\partial w \partial v}_{|v=w=0} L(v,w) = &\int_a^b \left(\langle \nabla_T V^\perp, \nabla_T W^\perp\rangle - \langle R(W^\perp, T)T, V^\perp\rangle\right) dt \\ &+ \langle \nabla_W V, T\rangle|_a^b,\end{aligned}$$

5. (Existence of Riemmannian metrics with positive curvature)
Consider the smooth manifold $RP^n \times RP^n$.

(i) Does there exist a Riemannian metric on $RP^n \times RP^n$ with positive sectional curvature?
(ii) Does there exist a Riemannian metric on $RP^n \times RP^n$ with positive Ricci curvature?

Remark: It is completely unknown whether $S^2 \times S^2$ has a Riemannian metric with positive sectional curvature or not. This is known as the Hopf problem.