## § $2\frac{1}{2}$ Cut point and Cut locus.

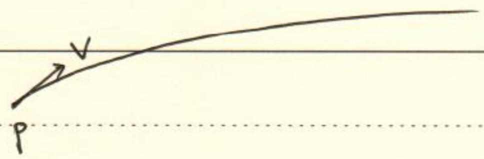
We concern the following two basic questions:

① Is a shortest curve parametrized by arc length a geodesic?

② Is a geodesic the shortest curve connecting its end points?

The answer to ① is yes by the local existence and uniqueness of shortest curves and the fact that every "subcurve" of a shortest one is again shortest.

However, the answer to ② is not definite. We know from previous discussion that a geodesic is shortest when its length is small enough. When the length is large, a geodesic can be not shortest anymore.



~~That is~~. ~~For~~ any $p \in M$, and $V \in T_pM$ with $\|V\|=1$, suppose the ~~the~~ geodesic

$\gamma(t) := \exp_p tV$

is defined on a maximal interval $[0,b)$, $b \in \mathbb{R}_+$ or $b = \infty$.

Recall that the interval is open at the $b$ end ~~since~~ is due to the local existence and uniqueness of geodesics.

Observe that if $\gamma|_{[0,t]}$ is shortest, then so is $\gamma|_{[0,t']}$ for any $t' \in (0,t)$. That is, if we set

$$A := \{t > 0 : d(p, \gamma(t)) = t\}$$
$$= \{t > 0 : \gamma|_{[0,t]} \text{ is shortest}\}$$

Then either $A = (0, b)$ or $A = (0, a]$ for some $0 < a < b$.
Notice that, the latter case means $\gamma|_{[0,t]}$ is not shortest for all $t > a$.

- If $A = (0, a]$ for some $0 < a < b$, we say that $\gamma(a)$ is the <u>cut point</u> of $p$ along the geodesic $\gamma$.

- If $A = (0, b)$, we say that ~~$\gamma(a)$~~ $p$ has no cut point along $\gamma$.

<u>Definition</u> (Cut locus) The <u>cut locus</u> $C(p)$ ~~of~~ $\subset M$ of $p$ is the set of all points which are cut points of $p$ along some normal geodesic eminating from $p$.
We also call the <u>cut locus</u> $\tilde{C}(0)$ of $0 \in T_pM$ to be the set of all vectors $aX \in T_pM$ for which $X$ is a unit tangent vector and $\exp_p(aX)$ is the cut point of $p$ along the geodesic $\gamma_X(t) := \exp_p(tX)$.

<u>Remark</u>. We have then $\exp_p(\tilde{C}(0)) = C(p)$

<u>Example</u>: ~~a~~) Open disc in $E^2$ with center $p$ and radius $r$.
Then $p$ has no cut points, i.e. $C(p) = \phi$. In fact, for any $V \in T_pM$, $\|V\|=1$, and the corresponding $\gamma(t) := \exp_p(tX)$, we have $A = (0, r)$.

b) Standard unit round sphere: the cut locus $C(p)$ is ~~the~~ its antipodal $p'$. In the tangent space $T_pM$, we have $\tilde{C}(0) = \partial B(0, \pi)$.

## §3½ Structure of complete Riemannian manifolds

We first observe that the exponential map $\exp_p$ at any point $p$ of a complete Riemannian manifold is surjective. by Hopf-Rinow theorem. ~~Since~~ Indeed, Completeness ensures that there exists a shortest curve from $p$ to any ~~other~~ point $q \in M$. That is, every $q \in M$ possesses a preimage under $\exp_p$ in ~~$C_p \subset T_pM$~~ $T_pM$. If we denote the shortest curve from $p$ to $q$ by the normal geodesic $\gamma(t)$ with $\gamma(0)=p$, then the preimage of $q$ is just ~~$\dot\gamma(0) \in T_pM$~~. $d(p,q)\cdot\dot\gamma(0) \in T_pM$.

Recall that $\exp_p$ is always a local diffeomorphism. In general, we cannot expect ~~that~~ $\exp_p$ to be a global diffeomorphism. That is, we care about "how global" this diffeomorphism can be. More precisely, we hope to ask

- How large can the Normal neighborhood of $p$ be?

Let us first recall the following useful corollary of the inverse function theorem.

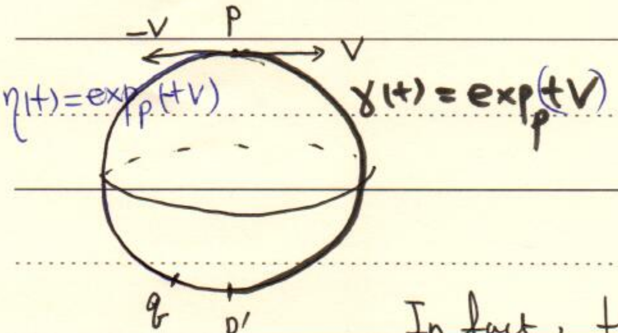
[See BSSG pp.11] For any $U \subset \mathbb{R}^n$, a $C^k$ mapping $F: U \to F(U) \subset \mathbb{R}^n$ is a $C^k$ diffeomorphism iff it is one-to-one and $dF$ is nonsingular at any point of $U$.

That is, to check a $C^k$ map to be a diffeomorphism, we only need to check its "local one-to-one" and "global one-to-one". The global 1-1 ensures the existence of the inverse map, and the nonsingularity of $dF$ ensures the smoothness of the inverse map.

Remark. Notice that without the nonsingularity of $dF$, the conclusion is not true anymore. A counterexample can be $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x^3$. It is $C^\infty$, 1-1 but not a diffeomorphism.

Before the general discussion, let us first investigate some examples.

Example 1 Round Sphere.



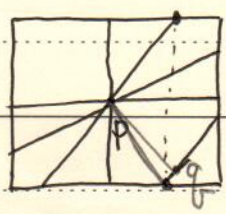

Cut locus of $p$ : $C(p) = \{p'\}$

Moreover: $\exp_p : B(0,\pi) \subset T_pM \to S^2 \setminus C(p)$ is a diffeomorphism.

In fact, this map can be expressed explicitly as: $tV \longmapsto \cos(t\|V\|)\, p + \sin(t\|V\|) \frac{V}{\|V\|}$

With this, one can check the nonsingularity of $d\exp_p$ at any point of $B(0,\pi) \setminus \{0\}$. Recall the nonsingularity of $d\exp_p$ at $0$ is known. The global 1-1 property of $\exp_p : B(0,\pi) \subset T_pM \to S^2 \setminus C(p)$ is clear. Therefore it is a diffeomorphism.

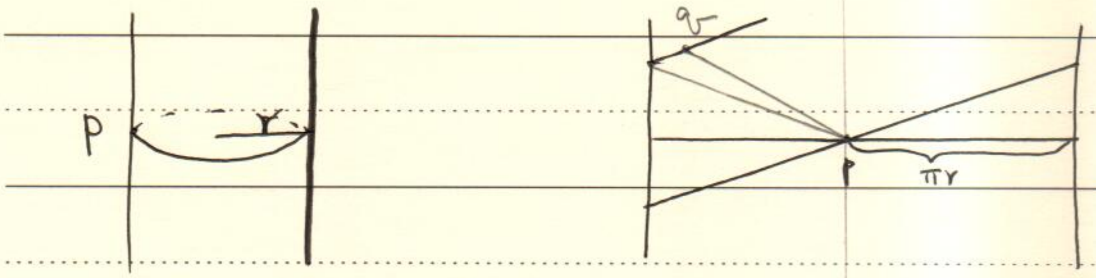
Example 2 Flat torus




Cut locus of $p$ : $C(p)$ = the boundary of the rectangle under identification.

Intuitively, $\exp_p$ is a diffeomorphism from this open rectangular region in $T_pM$ ~~and~~ to $M \setminus C(p)$.

Example 3. Cylinder



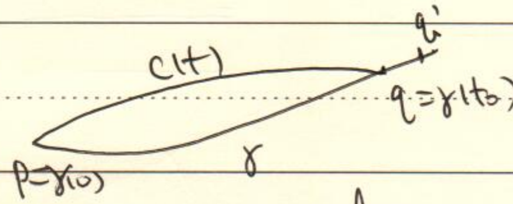

The cut locus of $p$ : $C(p)$ = The vertical line passing its antipodal.

The exponential map $\exp_p$ is a diffeomorphism from The "belt region" in $T_pM$ to the Cylinder $\setminus C(p)$.

~~The above examples tell us the cut locus seems to be the obstacle of the further extension of normal neighborhood.~~

~~Indeed it~~

Remark (8) The cut points in the above examples have the same property that there exist two shortest curves from $p$ to a point in $C(p)$. We can show this phenomena generally ~~implies~~ means cut point. That is, for a given geodesic $\gamma(t)$, if there are two shortest curves from $p = \gamma(0)$ and $q = \gamma(t_0)$, then ~~$q$ is a cut point~~ ~~all the points~~ $\gamma|_{[0,t]}$ is not shortest anymore for any $t > t_0$.




Let us denote the other shortest curve as $c(t)$. Then suppose $\gamma|_{[0,t']}$ is still shortest from $p$ to $q' = \gamma(t')$, then we have another path $\zeta : [0, t'] \to M$

$$\zeta(t) = \begin{cases} c(t), & t \in [0, t_0] \\ \gamma(t), & t \in [t_0, t'] \end{cases}.$$

which is of the same length as $\gamma|_{[0,t']}$, but not smooth at $t_0$. (By uniqueness of geodesics).
This contradicts to the fact that any shortest curve is smooth. ⊗

If moreover, $t_0$ is the smallest value that such phenomena happens, then $q$ is indeed a cut point. □

The above examples provide hints ~~that~~ cut locus might be the obstacle ~~of~~ to the further extension of normal neighborhood. Indeed, we have.

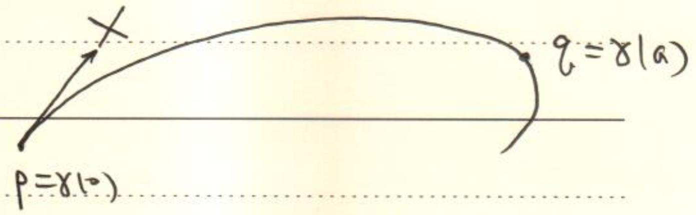
Theorem 1 (Cut points) Let $(M, g)$ be complete. and let $\gamma : [0, \infty) \to M$ be a normal geodesic with $p = \gamma(0)$ and cut point $\gamma(a)$. Then at least one of the following holds:

(i) $d\exp_p$ is singular at $a\dot{\gamma}(0) \in T_pM$.

(ii) There are at least two shortest curves from $p = \gamma(0)$ to $q = \gamma(a)$, and $a$ is the minimal value such that this case happens.

Remark. Theorem ~~(ii)~~ means $\exp_p$ is not injective on $\tilde{C}(0) \subset T_pM$.
~~Our previous.~~

Proof:




Denote by $X := \dot{\gamma}(0)$ the unit tangent vector of $\gamma$.

Then $\gamma(t) := \exp_p(tX)$.

Choose a sequence of numbers "after $a$":

$$a_1 > a_2 > a_3 > \cdots \quad \text{such that} \quad \lim_{i\to\infty} a_i = a.$$

By definition of a cut point, we know $b_i := d(p, \gamma(a_i)) < a_i$

By completeness, there are unit tangent vectors $X_i \in T_pM$ s.t. $\gamma_i(t) := \exp_p(tX_i)$, $t \in [0, b_i]$ is the shortest curve from $p$ to $\gamma(a_i) = \exp_p(b_iX_i)$

Since $\gamma(a)$ is a cut point of $p$ along $\gamma$, we know ~~every~~ all $X_i$'s are distinct from $X$.

By continuity of distance functions, we have

$$\lim_{i\to\infty} b_i = \lim_{i\to\infty} d(p, \gamma(a_i)) = d(p, \gamma(a)) = a.$$

Therefore $\{b_iX_i\}_{i=1}^{\infty}$ is contained in a compact subset of $T_pM$.

Choosing a subsequence if necessary, we have

$$\lim_{i\to\infty} b_iX_i = aY \quad \text{for some unit vector } Y \in T_pM.$$

Observing that $\exp_p(aY) = \lim_{i\to\infty} \exp_p(b_iX_i) = \lim_{i\to\infty} \gamma(a_i) = \gamma(a)$, we see the geodesic $t \mapsto \exp_p(tY)$, $t \in [0, a]$ is a curve from $p$ to $\gamma(a)$ of length $a$, i.e., it is ~~the~~ a shortest one.

Case (1) If $X \neq Y$, we have two shortest curves from $p = \gamma(0)$ to $q = \gamma(a)$. By the arguments from the previous Remark 4, we obtain that for any $t \in (0, a)$, There exists a unique shortest curve from $p$ to $\gamma(t)$. That is, $a$ is the minimal value such that more than one shortest curve can happen.

<u>Case (2)</u> If $X = Y$, we have

$$\lim_{i\to\infty} b_i X_i = aY = aX = \lim_{i\to\infty} a_i X.$$

On the other hand, we have

$$\exp_p(b_i X_i) = \gamma(a_i) = \exp_p(a_i X).$$

Moreover, $b_i X_i$ and $a_i X$ are distinct for each $i$, since $b_i < a$, $X_i \neq X$. Therefore, $\exp_p$ is not 1-1 in any neighborhood of $aX \in T_pM$. That is, $aX \in T_pM$ is a <u>critical point</u> of $\exp_p$. In other words, $d\exp_p$ is singular at $aX = a\dot{\gamma}(0)$. □

Now we have the following important observation: Let us ~~Theorem: Let $(M^n, g)$ be complete~~. first introduce some notations.

For $p \in M$, denote by $S_p$ the unit sphere of $T_pM$. Let $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$. We define a function

$$\tau : S_p \longrightarrow \mathbb{R}^*$$

as follows

$$\tau(X) = \begin{cases} a, & \text{if } \exp_p(aX) \text{ is the cut point of } p \text{ along the geodesic } t \mapsto \exp_p(tX). \\ \infty, & \text{if } p \text{ has no cut point along the geodesic } t \mapsto \exp_p(tX). \end{cases}$$

We further define

$$E(p) := \{ tV : V \in S_p \text{ and } 0 \le t < \tau(V) \} \subset T_pM.$$

<u>Theorem 2</u> Let $(M^n, g)$ be complete. Then $M$ is the disjoint union of $\exp_p(E(p))$ and $C(p)$, i.e.

$$M = \exp_p(E(p)) \sqcup C(p).$$

<u>Proof</u>: By completeness, $\exp_p$ is surjective on

$$E(p) \cup \tilde{C}(O_p) \subset T_pM.$$

That is, $M \subset \exp_p(E(p)) \cup C(p)$.

On the other hand, ~~for~~ ~~D~~ it is clear $\exp_p(E(p)) \cup C(p) \subset M$.

Therefore we have

$$M = \exp_p(E(p)) \cup C(p).$$

Next, we show $\exp_p(E(p)) \cap C(p) = \emptyset$.

If not, let $W \in E(p)$ and $V \in \tilde{C}(O_p)$ satisfy

$$\exp_p W = \exp_p V =: q$$

<u>Case 1</u> If $\|V\| < \|W\|$, this means the geodesic $t \mapsto \exp_p(tW)$ $t \in [0,1]$ is not a shortest curve from $p$ to $q$, which contradicts to the assumption $W \in E(p)$.

<u>Case 2</u> If $\|V\| > \|W\|$, this means the geodesic $t \mapsto \exp_p(tV)$, $t \in [0,1]$ is not a shortest curve from $p$ to $q$, which contradicts to the assumption that $V \in \tilde{C}(O_p)$.

<u>Case 3</u> If $\|V\| = \|W\|$, then there are two shortest curves from $p$ to $q$. By the arguments of Remark (✱), we obtain that $q$ ~~is also the cut~~ lies "after" the cut point of $p$ on the geodesic $t \mapsto \exp_p tW$. This contradicts to the ~~fa~~ assumption that $W \in E(p)$.

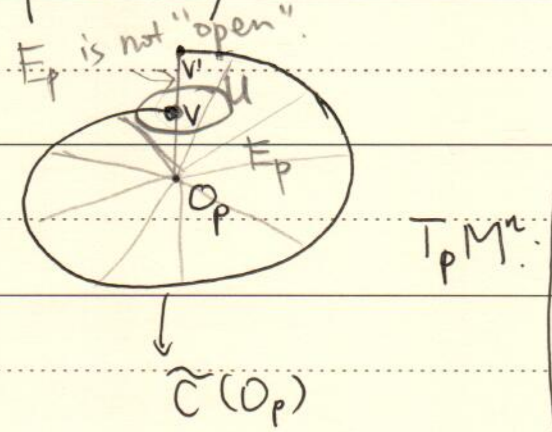
Therefore, our assumption is wrong. $\square$

Notice that $\exp_p$ is 1-1 on $E(p)$ (Using Rmk 4).
(injective).
~~That is~~ Moreover, by Theorem 1, $d\exp_p$ is nonsingular everywhere in $E(p)$. So by [BSSG pp.11], we know $\exp_p$ is a diffeomorphism on $E(p)$ if $E(p) \subset T_pM \cong \mathbb{R}^n$ is open.

Question. Is $E(p)$ always open?

One can check the examples of round sphere, flat torus, any cylinder; In those cases $E(p)$ are always open.

However, in general, we can not immediately exclude the following possibility:



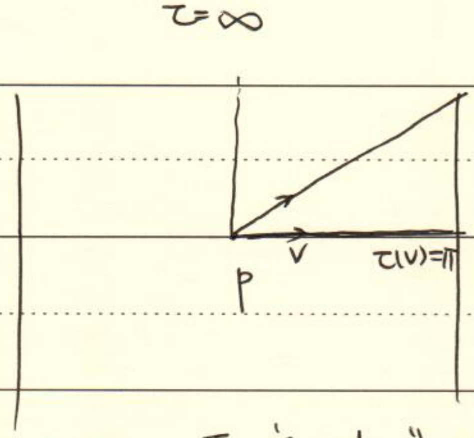

A rough idea why the jump at $V$ cannot happen:
If this happen, then $v$ lies inside $[O_p, V']$. Therefore, $d\exp_p$ is nonsingular at $V \Rightarrow \exists U \ni V$ s.t. $\exp_p|_U$ is a diffeomorphsim. Then the part of $\tilde{C}(O_p)$ in $U$ must be mapped to cut points which possess at least 2 shortest curves, In limit, this forces $\exp_p V$ has 2 shortest curves, contradiction!

Notice that, the above case can be excluded if the function $\tau: S_p \to \mathbb{R}^+$ is continuous! If it is continuous, then for any Cauchy ~~limit~~ sequence $\{\tau(V_i)V_i\}_{i=1}^{\infty}$ we have $\{V_i\}_{i=1}^{\infty}$ is Cauchy and $\lim_{i\to\infty} V_i = V$. Then

$$\lim_{i\to\infty} \tau(V_i)V_i = \tau(V)V.$$

That is, $\tilde{C}(O_p)$ is closed. Therefore $E_p$ is open!

However, we should be more careful: Consider the case of cylinder, we have $\tilde{C}(O_p)$




In this case, $\tau$ is not "continuous", but if we include $\infty$ by $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ and put the topology with a basis consisting of all sets of the form $(a,b) \subset \mathbb{R}$ together with all sets of the form $(a, \infty] := (a, \infty) \cup \{\infty\}$, then $\tau$ is ~~ag~~ continuous.

Theorem 3. Let $(M^n, g)$ be complete and $p \in M$. Then the function $\tau : S_p \longrightarrow \mathbb{R}^*$ is continuous.

Proof. Let $X_1, X_2, \cdots$ be a sequence of vectors in $S_p$ converging to $X \in S_p$. We aim to show $\{\tau(X_i)\}$ converges to $\tau(X)$.

Since ~~ε(Xi~~ all the values of $\tau$ lie in the compact set $\{\alpha \in \mathbb{R}^* : \alpha \geq 0\}$, $\{\tau(X_i)\}$ has convergent subsequences. Pick any subsequence, which we denote still by $\{\tau(X_i)\}$, and denote its limit by $\alpha \in \mathbb{R}^*$. We compare $\tau(X)$ and $\alpha$.

Case 1. If $\alpha = \infty$.

For any given $t$, there exists $N$ s.t. $\tau(X_i) > t$, $\forall i \geq N$.

Observe that

$$d(p, \exp_p(tX)) = \lim_{i \to \infty} d(p, \exp_p(tX_i)) \underset{\uparrow}{=} t$$

Since $\tau(X_i) > t$ when $i$ large enough.

That is, ~~t ⇒ exp~~ $\tau(X) \geq t$.

Since this holds for arbitrary given $t$, we have $\tau(X) = \infty$

<u>Case 2</u>: $\alpha < \infty$.

That is, $\lim_{i\to\infty} \tau(X_i) = \alpha$, and hence $\lim_{i\to\infty} \tau(X_i)X_i = \alpha X$.

$$d\big(p, \exp_p(\alpha X)\big) = \lim_{i\to\infty} d\big(p, \exp_p(\tau(X_i)X_i)\big)$$

$$= \lim_{i\to\infty} \tau(X_i) = \alpha.$$

By definition, this means

$$\tau(X) \geq \alpha.$$

In conclusion, for any convergent subsequence $\{\tau(X_i)\}$ with limit $\alpha \in \mathbb{R}^*$, we have $\tau(X) \geq \alpha$.

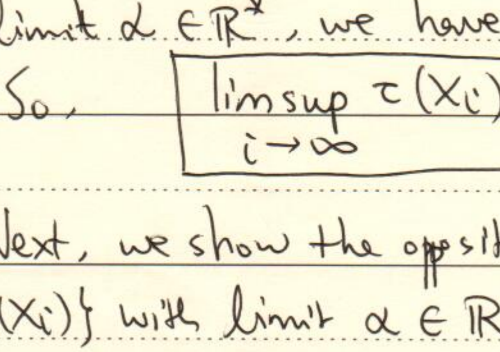
So, $\boxed{\limsup_{i\to\infty} \tau(X_i) \leq \tau(X)}$ (1).

Next, we show the opposite one: for any convergent subsequence $\{\tau(X_i)\}$ with limit $\alpha \in \mathbb{R}^*$, we have $\tau(X) \leq \alpha$.

We show it by contradiction. A

Suppose $\tau(X) > \alpha$. Then $d\exp_p$ is not singular at $\alpha X$. A

Therefore, $\exp_p$ is a diffeomorphism on a neighborhood $U$ of $\alpha X$ in $T_pM$. Since $\tau(X_i)X_i \to \alpha X$, we may assume all




$$\tau(X_i)X_i \in U.$$

Hence $d\exp_p$ is not singular at $\tau(X_i)X_i$, $\forall i$.

Then definition of $\tau(X_i)$ and Theorem 1 implies the existence of another shortest curve from $p$ to $\exp_p \tau(X_i)X_i$.

That is, there exists $Y_i \in S_p$, s.t. $\exp_p \tau(X_i)Y_i = \exp_p \tau(X_i)X_i$.

Since $\exp_p|_U$ is 1-1, we have $\tau(X_i)Y_i \notin U$.

By taking a subsequence when necessary, we assume
$$\lim_{i\to\infty} Y_i = Y \in S_p .$$
Then we have $\lim_{i\to\infty} \tau(X_i)Y_i = \alpha Y$.
Since $U$ is open, $\tau(X_i)Y_i \notin U$, we have $\alpha Y \notin U$.
In particular, $\alpha Y \neq \alpha X$.
However, $\exp_p(\alpha Y) = \lim_{i\to\infty} \exp_p \tau(X_i)\cancel{Y_i} = \lim_{i\to\infty} \exp_p \tau(X_i) X_i$
$$= \exp_p(\alpha X).$$
That is, we find two shortest curves from $p$ to $\exp_p(\alpha X)$.
This contradicts to the assumption that $\tau(X) > \alpha$.
Therefore, our assumption is wrong, and we have $\tau(X) \leq \alpha$.
This means
$$\boxed{\tau(X) \leq \liminf_{i\to\infty} \tau(X_i)} \quad (2)$$
Combining (1) and (2), we ~~finish~~ arrive at
$$\tau(X) = \lim_{i\to\infty} \tau(X_i) .$$
D.

Therefore, by previous Remarks, we derive that $E(p)$ is open in $T_pM$. And by [BSSG pp.11], we see (B)
$$\exp_p : E_p \hookrightarrow \exp_p(E_p) = M \setminus C(p)$$
is a diffeomorphism!!

Conclusion: What is the cut locus of a point $p$ in a complete Rie. mfld?
It is a closed subset $C(p)$ of $M$ such that, after "cutting the mfld along it," $M \setminus C(p)$ becomes a normal neighborhood!

On page 62 + 10 (claim A) and page 62 + 11 (claim B), We still need the following result: (comparing with Theorem 1):

<u>Theorem 4</u>. Let $(M^n, g)$ be complete, and $p \in M$. ~~Then~~ If $d\exp_p$ is singular at $aX \in T_pM$, $X \in S_p$,

Then $a \geq \tau(X)$. Equivalently to say, if $a < \tau(X)$, then $d\exp_p$ is nonsingular at $aX \in T_pM$.