

(IV) Curvature

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Why is the curvature tensor R called "curvature"? This is the question that we're going to explain in this chapter.

The local expression

$$R^k_{lij} = \frac{\partial \Gamma^k_{il}}{\partial x^j} - \frac{\partial \Gamma^k_{jl}}{\partial x^i} + \Gamma^k_{jl} \Gamma^i_{ir} - \Gamma^k_{il} \Gamma^j_{jr}$$

appeared already in an unpublished paper of Riemann submitted to the Paris Academy in 1861. (See [Spirak II, pp. 181]) (This happened before the birth of Christoffel symbols).

The curvature tensor R has its origin in the following question, which Riemann considered in the 1861 paper: In a chart (U, θ) of a Rie. mfd (M, g) , the metric tensor can be formulated as

$$g_{ij} dy^i \otimes dy^j, \quad (*)$$

~~are there~~ Can we find a coordinate transformation

$$x^\alpha = x^\alpha(y^1, \dots, y^n), \quad \alpha=1, \dots, n$$

such that ~~$g_{ij} dy^i \otimes dy^j = g'_{\alpha\beta} dx^\alpha \otimes dx^\beta$~~ $(*)$ is transformed into $g'_{\alpha\beta} dx^\alpha \otimes dx^\beta$

for some given functions $(g'_{\alpha\beta})_{\alpha, \beta=1, \dots, n}$, (of course, we require the matrix $(g'_{\alpha\beta})$ to be symmetric and positive definite). This is equivalent to ask when are two Rie. mfd's locally isometric?

We are actually asked to find functions $x^\alpha = x^\alpha(y^1, \dots, y^n)$, $\alpha=1, \dots, n$ such that

$$g_{ij}(y) = g'_{\alpha\beta}(x(y)) \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j}, \quad i, j=1, \dots, n \quad (**)$$

That is, we need solve the above PDE. Let us denote by

$$\Gamma^k_{jk}, \quad \Gamma'^\lambda_{\alpha\beta}$$

the Christoffel symbols w.r.t. (g_{ij}) and $(g'_{\alpha\beta})$ respectively. Recall that, if such coordinate transformation exists, we have

$$\Gamma_{jk}^p = \Gamma'_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \frac{\partial y^p}{\partial x^\lambda} + \frac{\partial^2 x^\mu}{\partial y^j \partial y^k} \frac{\partial y^p}{\partial x^\mu}$$

Therefore

$$\Gamma_{jk}^p \frac{\partial x^\lambda}{\partial y^p} = \Gamma'_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \delta_\gamma^\lambda + \frac{\partial^2 x^\mu}{\partial y^j \partial y^k} \delta_\mu^\lambda$$

That is,

$$(*) \quad \left[\frac{\partial}{\partial y^k} \left(\frac{\partial x^\lambda}{\partial y^j} \right) \right] = \Gamma_{jk}^p \frac{\partial x^\lambda}{\partial y^p} - \Gamma'_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \quad \lambda, j, k = 1, \dots, n.$$

So a necessary condition that $(*)$ has a solution is that the above PDE has a solution

$$\left(\frac{\partial x^\lambda}{\partial y^j} \right)_{\lambda, j=1, \dots, n}$$

By PDE theory, (See [Spivak I, pp. 187]), $(*)$ has a solution iff the following integrability condition holds:

$$(***) \quad \frac{\partial}{\partial y^l} \frac{\partial}{\partial y^k} \left(\frac{\partial x^\lambda}{\partial y^j} \right) = \frac{\partial}{\partial y^k} \frac{\partial}{\partial y^l} \left(\frac{\partial x^\lambda}{\partial y^j} \right), \quad \forall \lambda, l, k, j.$$

Claim, $(***)$ is equivalent to

$$\frac{\partial x^\alpha}{\partial y^k} \frac{\partial x^\beta}{\partial y^l} \frac{\partial x^\gamma}{\partial y^j} R'^{\lambda}_{\alpha\beta\gamma} = R^p_{j\ell k} \frac{\partial x^\lambda}{\partial y^p}, \quad \text{where } R', R \text{ are the curvature tensor w.r.t. } g', g.$$

Proof: By $(*)$, we calculate

$$\begin{aligned} \frac{\partial}{\partial y^l} \frac{\partial}{\partial y^k} \left(\frac{\partial x^\lambda}{\partial y^j} \right) &= \frac{\partial}{\partial y^l} \left(\Gamma_{jk}^p \frac{\partial x^\lambda}{\partial y^p} - \Gamma'_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \right) \\ &= \frac{\partial \Gamma_{jk}^p}{\partial y^l} \frac{\partial x^\lambda}{\partial y^p} + \Gamma_{jk}^p \frac{\partial}{\partial y^l} \left(\frac{\partial x^\lambda}{\partial y^p} \right) - \frac{\partial}{\partial y^l} (\Gamma'_{\alpha\beta}) \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \\ &\quad - \Gamma'_{\alpha\beta} \frac{\partial}{\partial y^l} \left(\frac{\partial x^\alpha}{\partial y^j} \right) \frac{\partial x^\beta}{\partial y^k} - \Gamma'_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial}{\partial y^l} \left(\frac{\partial x^\beta}{\partial y^k} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial \Gamma_{jk}^p}{\partial y^l} \frac{\partial x^\lambda}{\partial y^p} + \Gamma_{jk}^p \left(\Gamma_{pl}^q \frac{\partial x^\lambda}{\partial y^q} - \Gamma_{\alpha p}^{\lambda'} \frac{\partial x^\alpha}{\partial y^p} \frac{\partial x^\beta}{\partial y^l} \right) - \frac{\partial \Gamma_{\alpha p}^{\lambda'}}{\partial x^\delta} \frac{\partial x^\delta}{\partial y^l} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \\
&\quad - \Gamma_{\alpha p}^{\lambda'} \left(\Gamma_{jl}^q \frac{\partial x^\alpha}{\partial y^q} - \Gamma_{\delta l}^{\alpha'} \frac{\partial x^\delta}{\partial y^j} \frac{\partial x^\eta}{\partial y^l} \right) \frac{\partial x^\beta}{\partial y^k} - \Gamma_{\alpha p}^{\lambda'} \frac{\partial x^\alpha}{\partial y^j} \left(\Gamma_{kl}^q \frac{\partial x^\beta}{\partial y^q} - \Gamma_{\delta l}^{\beta'} \frac{\partial x^\delta}{\partial y^k} \frac{\partial x^\eta}{\partial y^l} \right) \\
&\stackrel{\square}{=} \left(\frac{\partial \Gamma_{jk}^p}{\partial y^l} + \Gamma_{jk}^p \Gamma_{pl}^q \right) \frac{\partial x^\lambda}{\partial y^q} - \Gamma_{jk}^p \Gamma_{\alpha p}^{\lambda'} \frac{\partial x^\alpha}{\partial y^p} \frac{\partial x^\beta}{\partial y^l} - \Gamma_{\alpha p}^{\lambda'} \Gamma_{jl}^q \frac{\partial x^\alpha}{\partial y^q} \frac{\partial x^\beta}{\partial y^k} \\
&\quad - \Gamma_{\alpha p}^{\lambda'} \Gamma_{kl}^q \frac{\partial x^\alpha}{\partial y^q} \frac{\partial x^\beta}{\partial y^l} - \frac{\partial \Gamma_{\alpha p}^{\lambda'}}{\partial x^\delta} \frac{\partial x^\delta}{\partial y^l} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \\
&\quad + \Gamma_{\alpha p}^{\lambda'} \Gamma_{\delta l}^{\alpha'} \frac{\partial x^\delta}{\partial y^j} \frac{\partial x^\eta}{\partial y^l} \frac{\partial x^\beta}{\partial y^k} + \Gamma_{\alpha p}^{\lambda'} \Gamma_{\delta l}^{\beta'} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\delta}{\partial y^k} \frac{\partial x^\eta}{\partial y^l}
\end{aligned}$$

Notice that the symmetry of Christoffel symbols implies that the terms " ν_3 " and " o " are symmetric w.r.t. k and l . Moreover, $\nu_1 + \nu_2$ is symmetric w.r.t. k and l .

Therefore, we have

$$\begin{aligned}
\frac{\partial}{\partial y^l} \frac{\partial}{\partial y^k} \left(\frac{\partial x^\lambda}{\partial y^j} \right) &= \left(\frac{\partial \Gamma_{jk}^p}{\partial y^l} + \Gamma_{jk}^p \Gamma_{pl}^q \right) \frac{\partial x^\lambda}{\partial y^q} \\
&\quad - \left(\frac{\partial \Gamma_{\alpha p}^{\lambda'}}{\partial x^\delta} - \Gamma_{\mu \beta}^{\lambda'} \Gamma_{\alpha \delta}^{\mu} \right) \frac{\partial x^\delta}{\partial y^l} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \\
&\quad + \text{terms which are symmetric w.r.t. } k \text{ and } l.
\end{aligned}$$

Therefore, $\frac{\partial}{\partial y^l} \frac{\partial}{\partial y^k} \left(\frac{\partial x^\lambda}{\partial y^j} \right) - \frac{\partial}{\partial y^k} \frac{\partial}{\partial y^l} \left(\frac{\partial x^\lambda}{\partial y^j} \right) = 0$ implies

$$\begin{aligned}
0 &= \left(\frac{\partial \Gamma_{jk}^p}{\partial y^l} - \frac{\partial \Gamma_{jl}^q}{\partial y^k} + \Gamma_{jk}^p \Gamma_{pl}^q - \Gamma_{jl}^q \Gamma_{pk}^q \right) \frac{\partial x^\lambda}{\partial y^q} \\
&\quad - \left(\frac{\partial \Gamma_{\alpha p}^{\lambda'}}{\partial x^\delta} - \frac{\partial \Gamma_{\alpha \delta}^{\lambda'}}{\partial x^\beta} + \Gamma_{\alpha \beta}^{\mu} \Gamma_{\mu \delta}^{\lambda'} - \Gamma_{\alpha \delta}^{\mu} \Gamma_{\mu \beta}^{\lambda'} \right) \frac{\partial x^\delta}{\partial y^l} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k}
\end{aligned}$$

That is $0 = R_{jlk}^q \frac{\partial x^\lambda}{\partial y^q} - R_{\alpha \delta \beta}^{\lambda'} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\delta}{\partial y^l} \frac{\partial x^\beta}{\partial y^k}$ \square