Soul Theorem and Soul Conjecture

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1 Background Information

It is a central question in Riemannian geometry about the structure of complete manifolds whose sectional curvature is of a fixed sign. Before Cheeger and Gromoll, most attention had been paid to compact case, with an exception that is the work of Cohn-Vossen. He actually obtained the following result:

Theorem (Cohn-Vossen)**.** *In dimension 2, a noncompact complete manifold of nonnegative curvature is either diffeomorphic to* $\mathbb{R}^2$ *or is flat.*

In 1972, Cheeger and Gromoll generalized Cohn-Vossen's result as follows:

Theorem (Soul Theorem)**.** *Let* M *be a complete noncompact Riemannian manifold with nonnegative sectional curvature. Then* M *contains a compact totally geodesic and totally convex embedding submanifold* S *without boundary,* $0 \leq dimS < dimM$*, such that* M *is diffeomorphic to the total space of the normal bundle of* S *in* M*.*

Finally in their paper, they asked a natural question whether the result of Cohn-Vossen still holds for higher dimension. This is called the **Soul Conjecture**: suppose M is complete and noncompact with sectional curvature ≥ 0, and there exists at least one point p such that the curvature is positive for all sections in T_pM. Is the soul of M always a point? Or equivalently, M is diffeomorphic to Euclidean space $\mathbb{R}^n$?

The soul conjecture holds if M has positive sectional curvature, due to an earlier result of Gromoll and Meyer. The case $n = 3$ was verified by Yu.D.Burago in 1979 and case $n = 4$ verified by V.B. Marenich in 1980. In 1992 and 1993, Marenich published an argument for analytic manifolds without dimensional restrictions, containing over 50 pages of computations.

Finally in 1994, G.Perelman presented an elegant proof of Soul Conjecture in 4 pages.

2 Perelman's proof of Soul Conjecture

The proof depends mainly on two results: the Berger's version of Rauch comparison theorem[3] and the existence of distance nonincreasing retraction of M onto S due to Sharafutdinov[4]. In fact, Perelman proved the following result:

Theorem (Perelman,1994)**.** *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature, let S be a soul of M, and let $P : M \to S$ be a distance nonincreasing retraction. Then:*

(A)For any $x \in S$,$\nu \in SN(S)$, we have

$$P(exp_x(t\nu)) = x \textit{ for all } t \geq 0$$

where $SN(S)$ denotes the unit normal bundle of S in M.

(B)For any geodesic $\gamma \subset S$ and any vector field $\nu \in \Gamma(SN(S))$ parallel along γ, the "horizontal" curves γ_t,$\gamma_t(u) = exp_{\gamma(u)}(t\nu)$, are geodesics, filling a flat totally geodesic strip($t \geq 0$). Moreover,if $\gamma[u_0, u_1]$ is minimizing, then all $\gamma_t[u_0, u_1]$ are also minimizing.

The Soul Conjecture is now a corollary of (B) since the normal exponential map is surjective. Therefore we can find a normal geodesic from a point in a soul S to the point p whose sectional curvature is always positive. If $dim(S)$>0, p will sit in a flat surface formed by geodesic strip, which contradicts the property of p.

Proof. We observe first that it is sufficient to show that if (A) and (B) hold for $0 \leq t \leq l$ for some $l \geq 0$, then they continue to hold for $0 \leq t \leq l + \epsilon(l)$, for some $\epsilon(l)$>0.

Suppose that (A) and (B) hold for $0 \leq t \leq l$. For small $r \geq 0$, consider the function

$$f(r) = max\{d(x, P(exp_x(l + r)\nu)) : x \in S, \nu \in SN_x(S)\}$$

Then $f(0) = 0$. And we contend that f is a Lipschitz function.Actually, we have:

$$\begin{aligned} &d(x, P(exp_x(l + r_1)\nu)) \\ &\quad \leq d(x, P(exp_x(l + r_2)\nu)) + d(P(exp_x(l + r_2)\nu), P(exp_x(l + r_1)\nu)) \\ &\qquad \leq f(r_2) + d(exp_x(l + r_2)\nu, exp_x(l + r_1)\nu) \\ &\qquad\qquad \leq f(r_2) + |r_1 - r_2| \end{aligned}$$

which implies $f(r_1) \leq f(r_2) + |r_1 - r_2|$. Similarly, we can prove the other side and show that f is Lipschitz continuous.

We are going to show that the upper left derivative of f is nowhere positive for small r. It follows that f is non-increasing. Since $f \geq 0$ and $f(0) = 0$, f is identically zero for small r.

Fix r>0. Let $f(r) = d(x_0, \bar{x_0})$ where $\bar{x_0} = P(exp_{x_0}(l + r)\nu_0)$. Since r is small, and f is continuous, we may assume that $f(r)$<$injrad(S)$. Pick a point $x_1 \in S$ so that x_0 lies on a minimizing geodesic between $\bar{x_0}$ and x_1; let $x_0 = \gamma(u_0)$,$x_1 = \gamma(u_1)$. Let $\nu \in \Gamma(SN(S))$ be a parallel vector field along γ, $\nu_{x_0} = \nu_0$. Then according to our assumption, the curve $\gamma_t(u) = exp_{\gamma(u)}(t\nu)$,$0 \leq t \leq l$, are minimizing geodesics of filling a flat totally geodesic rectangle.

It follows our assumption that:

1. The tangent vector field of normal geodesics is parallel along "horizontal" geodesic;
2. The tangent vector field of "horizontal" geodesics is parallel along normal geodesic.

In fact, denote by $F(t,u) = exp_{\gamma(u)}(t\nu)$, we have:

$$\frac{D}{\partial t}\frac{D}{\partial u}\frac{\partial F}{\partial t} = \frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial F}{\partial t} + R(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial t})\frac{\partial F}{\partial t} = 0$$

Hence

$$\frac{D}{\partial u}\frac{\partial F}{\partial t} = \frac{D}{\partial u}\frac{\partial F}{\partial t}(u,0) = 0$$

which proves the first assertion. By torsion-free, the second holds.In particular, the normal geodesic and "horizontal" geodesic are orthogonal everywhere; and all "horizontal" geodesics are of constant length.

According to Berger's comparision theorem, the arcs of γ_{l+r} are no longer than corresponding arcs of γ_l, with equality if and only if $\gamma_t, l \leq t \leq l+r$, are geodesics filling a flat totally geodesic rectangle. Now consider the point $\bar{x_1} = P(exp_{\gamma(u_1)}(l+r)\nu)$. Using the distance decreasing property of P and the above observation we get:

$$\begin{aligned} d(\bar{x_0}, \bar{x_1}) &\leq d(exp_{\gamma(u_0)}(l+r)\nu, exp_{\gamma(u_1)}(l+r)\nu) \\ &\leq L(\gamma_{l+r}[u_0, u_1]) \\ &\leq L(\gamma_l[u_0, u_1]) = d(x_0, x_1) \end{aligned}$$

On the other hand,

$$d(x_1, \bar{x_1}) \leq f(r) = d(x_0, \bar{x_0})$$

Taking into account that by construction

$$d(x_0, \bar{x_0}) + d(x_0, x_1) = d(\bar{x_0}, x_1) \leq d(x_1, \bar{x_1}) + d(\bar{x_0}, \bar{x_1})$$

We see then the above must be equalities, and therefore

$$\gamma_t[u_0, u_1], \ l \leq t \leq l+r$$

are minimizing geodesics filling a flat totally geodesic rectangle.

Now let $\delta \to 0$, we obtain

$$\begin{aligned} f(r-\delta) &\geq d(x_1, P(exp_{x_1}(l+r-\delta)\nu)) \\ &\geq d(\bar{x_0}, x_1) - d(\bar{x_0}, P(exp_{x_1}(l+r-\delta)\nu)) \\ &\geq d(\bar{x_0}, x_1) - d(exp_{x_1}(l+r-\delta)\nu, exp_{x_0}(l+r)\nu) \end{aligned}$$

By Toponogov's comparision theorem,

$$\begin{aligned} & d(exp_{x_1}(l+r-\delta)\nu, exp_{x_0}(l+r)\nu) \\ &\leq \sqrt{d(exp_{x_1}(l+r)\nu, exp_{x_0}(l+r)\nu) + \delta^2} \\ &= d(exp_{x_1}(l+r)\nu, exp_{x_0}(l+r)\nu) + O(\delta^2) \end{aligned}$$

Hence we have:

$$
\begin{aligned}
f(r-\delta) \geq d(\bar{x_0}, x_1) - d(exp_{x_1}(l+r)\nu, exp_{x_0}(l+r)\nu) - O(\delta^2) \\
= d(\bar{x_0}, x_1) - d(x_0, x_1) - O(\delta^2) \\
= d(\bar{x_0}, x_0) - O(\delta^2) = f(r) - O(\delta^2)
\end{aligned}
$$

which shows that the upper left derivative of f is nowhere positive, whence $f(r) \equiv 0$ for $0 \leq r \leq \epsilon(l)$. Therefore, (A) is proved for $0 \leq r \leq \epsilon(l)$. To prove (B) for such t, one can repeat a part of argument above, taking into account that (x_0, ν_0),γ,x_1 can now be chosen arbitrarily since $\bar{x_0} = x_0$,$\bar{x_1} = x_1$. □

3 References

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