§ Topological Sphere Theorem. 1.

In this section, we discuss the proof of the following fasinating theorem.

Theorem 0. Let M^n be a compact simply connected Riemannian manifold, whose sectional curvature K satisfies

$$0 < h K_{max} < K \leq K_{max} \qquad (1)$$

Then if $h = 1/4$, M is homeomorphic to a sphere.

The number h is called the "pinching" of M. By scaling, we can always suppose $K_{max} = 1$. Then (1) can be written as

$$0 < h < K \leq 1.$$

History:

(1) Rauch 1951. All dimension $h = \frac{3}{4}$ $\left(\begin{array}{l}\text{solution of} \\ \sin(\pi\sqrt{h}) = \frac{\sqrt{h}}{2}\end{array}\right)$

A contribution to differential geometry in the large, Ann. of Math. 54 (1951), 38-55.

(2) Klingenberg 1959: even dimension, $h = 0.55$ $\left(\begin{array}{l}\text{solution of} \\ \sin(\pi\sqrt{h}) = \sqrt{h}\end{array}\right)$

Contributions to Riemannian geometry in the large, Ann. of Math. 69 (1959), 654-666.

(3) Berger 1960, even dimension, $h = \frac{1}{4}$.

Les variétés Riemannienes (1/4) - pincées, Ann. Scuola Norm. Sup. Pisa, Ser. III, 14 (1960), 161-170.

(4) Klingenberg 1961. All dimension, $h = \frac{1}{4}$

Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung, Comm. Math. Helv. 35 (1961), 47-54

Remark on low dimension cases

(1) $n=2$. Thm 1 holds for any $h \geq 0$.

This is due to Gauss-Bonnet formula.

(2) $n=3$. Thm 1 holds for any $h \geq 0$.

Hamilton 1982: Let (M^3, g) be a compact, connected, 3-dim Rie. mfld with $Ric > 0$. Then g can be deformed in the class of metrics with $Ric > 0$ into a metric with constant sectional curvature K_M. (Ricci flow method)

Therefore, any compact, simply connected 3-mfld M^3 with $Ric > 0$ is diffeomorphic to S^3.

§1. Cut locus and injectivity radius

Let us recall our previous discussions about cut locus of complete Rie. mflds.

Theorem 1. Let (M^n, g) be complete and let $\gamma: [0, \infty) \to M$ be a normal geodesic with $p = \gamma(0)$ and cut point $\gamma(a)$. Then at least one of the following holds.

(i) $\gamma(a)$ is the first conjugate point of $p = \gamma(0)$ along γ.

(ii) There are at least two shortest curves from $p = \gamma(0)$ to $q = \gamma(a)$, and a is the minimal value such that this case happens.

For any $p \in M$, denote by S_p the unit sphere of T_pM. let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. We define the function

$$\tau : S_p \longrightarrow \mathbb{R}^*.$$

as

$$\tau(X) = \begin{cases} a, & \text{if } \exp_p(aX) \text{ is the cut point of } p \text{ along the geodesic } t \mapsto \exp_p(tX) \\ \infty, & \text{if } p \text{ has no cut point along the geodesic } t \mapsto \exp_p(tX). \end{cases}$$

<u>Thm2</u>. Let (M^n, g) be complete and $p \in M$. Then the function $\tau : S_p \longrightarrow \mathbb{R}^*$ is continuous, where $\mathbb{R}^*$ is assigned with the topology with a basis consisting of all sets of the form $(a,b) \subset \mathbb{R}$ together with all sets of the form $(a, \infty] := (a, \infty) \cup \{\infty\}$

Define $E(p) := \{ tV : V \in S_p, 0 \leq t < \tau(V) \} \subset T_pM$.

<u>Thm3</u>. Let (M^n, g) be complete, $p \in M$. Then M is the disjoint union of $\exp_p(E(p)) \sqcup C(p)$. Moreover, $\exp_p$ maps $E(p)$ diffeomorphically onto an open subset of M.

<u>Definition</u> (Injectivity radius). Let (M^n, g) be a Rie. mfld, $p \in M$. The injectivity radius of p is

$$i(p) := \sup \{ \rho > 0 ; \exp_p \text{ is a diffeomorphism on } B(0,\rho) \subset T_pM \}.$$

The <u>injectivity radius</u> of M is defined as.

$$i(M) := \inf_{p \in M} i(p).$$

<u>Remark</u>: Recall that $\exp_p$ is injective on $E(p)$ ...
(diffeomorphism)

In fact, we have $i(p) = \sup\{\rho > 0 : B(0,\rho) \subset E(p)\}$.

or $i(p) = d(p, C(p)) = \inf_{q \in C(p)} d(p, q)$.

Observe that, when (M^n, g) is compact, $p \in M$,
$\tau(V) < \infty$, $\forall V \in S_p$. This is because compactness of (M^n, g) implies boundedness of its diameter. Therefore, by the continuous of τ and compactness of S_p, we have
$i(p)$ can be achieved. That is, $\exists\, q \in C(p)$ s.t.
$$d(p, q) = i(p).$$

In fact, one can extend Thm 2 to a bit more general form. Let T_1M be the <u>unit tangent bundle</u> of M and define the fct

$$\tilde{\tau} : T_1M \longrightarrow \mathbb{R}^* = \mathbb{R} \cup \{\infty\}$$

$$\tilde{\tau}(p, X_p) = \begin{cases} a, & \text{if } \exp_p(aX_p) \text{ is the cut pt of } p \text{ along the geodesic } t \mapsto \exp_p(tX_p) \\ \infty, & \text{if } p \text{ has no cut pt along the geodesic } t \mapsto \exp_p(tX_p). \end{cases}$$

<u>Theorem 2'</u> Let (M^n, g) be cpl. Then $\tilde{\tau} : T_1M \to \mathbb{R}^*$ is continuous.

<u>Proof</u>. Excercise. One can modify the proof of Thm 2.
([do Carmo] Prop. 2.9. pp. 272) □

<u>Remark</u>. Theorem 2' tells on a compact Rie. mfld. the $i(M)$ can be achieved: $\exists\ p, q \in M$ s.t. $q \in C(p)$

$$d(p, q) = i(M). \qquad (*)$$

This is because

$$i(M) = \inf_{p \in M} d(p, C(p)) = \inf_{p \in M} \inf_{q \in C(p)} d(p, q)$$

$$= \inf_{p \in M} \inf_{X_p \in S_p} \tilde{\tau}(p, X_p) = \inf_{(p, X_p) \in T_1M} \tilde{\tau}(p, X_p)$$

Then $(*)$ follows from the compactness of T_1M and continuity of $\tilde{\tau}$. □

§2. Injectivity Radius estimates.

Let (M^n, g) be complete, $p \in M$. We start by looking at the special property of $q \in C(p)$ with $d(p, q) = i(p)$.

<u>Thm 4</u>. Let (M^n, g) be complete, $p \in M$. Let $q \in C(p)$ s.t.

$$d(p, q) = i(p).$$

Then, at least one of the following holds.

(i) q is conjugate to p along some minimizing geodesic from p to q

(ii) there exists <u>exactly</u> two minimizing geodesic γ and σ from p to q; in addition, $\gamma'(l) = -\sigma'(l)$, $l = d(p, q)$.

<u>Proof</u>: Let γ be a minimizing geodesic from p to q. Then by Thm 1, either q is conjugate to p along γ (i.e., (i) holds)

Or $\exists$ another minimizing geodesic $\sigma \neq \gamma$, from p to q with $l(\sigma) = l(\gamma)$. We aim to show:

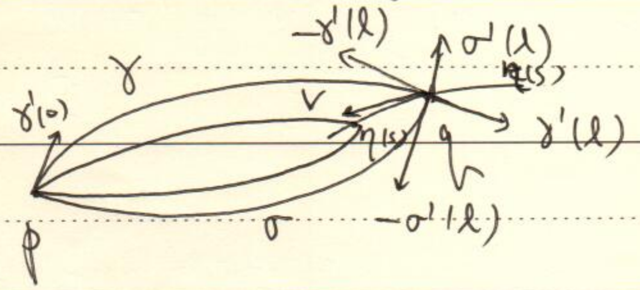
<u>Aim</u>: If q is not conj. to p along γ and σ, then

$$\gamma'(l) = -\sigma'(l). \qquad (*)$$

If <u>Aim</u> holds, then we see (ii) holds. (In fact, (*) shows there can be only two such geodesics).

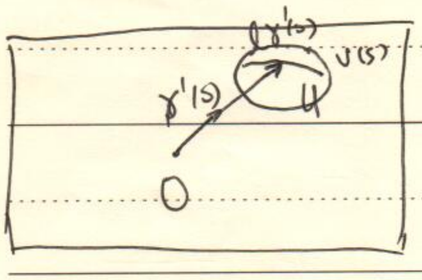
<u>Proof of Aim</u>: We do it by contradiction: Suppose

$$\gamma'(l) \neq -\sigma'(l).$$



Then $\exists\, V \in T_qM$ s.t.

$$\langle V, \gamma'(l) \rangle < 0$$
$$\langle V, \sigma'(l) \rangle < 0.$$



Let $\eta : (-\varepsilon, \varepsilon) \to M$ be a curve with $\eta(0) = q$, $\eta'(0) = V$.

Since q is not conjugate to p along γ, $\exists$ a neighborhood U of $l\gamma'(0)$ in T_pM, s.t. $\exp_p$ is a diffeomorphism on U.

Let $v : (-\varepsilon, \varepsilon) \to U$ be a curve s.t.

$$\exp_p v(s) = \eta(s), \quad s \in (-\varepsilon, \varepsilon).$$

We can then construct the variation:

$$F(s,t) = \exp_p \frac{t}{l} v(s), \quad t \in [0, l], \quad s \in (-\varepsilon, \varepsilon)$$

with $F(0,t) = \gamma(t)$.

The FVF for the length fct'l:

$$\left.\frac{d}{ds}\right|_{s=0} \text{Length}(s) = \langle V, \gamma'(l) \rangle < 0.$$

Observe further $F(s,l) = \exp_p v(s) = \eta(s)$.
If $s>0$ is small enough, $(\gamma_s(t) := F(s,t))$,

$$\text{Length}(\gamma_s) < \text{Length}(\gamma).$$

Similarly, since q is not conjugate to p along σ, we obtain a variation $\sigma_s(t)$ of σ, s.t.

$$\cancel{F(s,t) = \exp_p} \quad \sigma_s(l) = \eta(s)$$

and for $s>0$ small enough,

$$\text{Length}(\sigma_s) < \text{Length}(\sigma)$$

Case (1) $l(\gamma_s) = l(\sigma_s)$. Therefore, $\eta(s) = \gamma_s(l)$ is a cut point of p along γ_s (or σ_s). But

$$d(p, \gamma_s(l)) = l(\gamma_s) < d(p,q) = i(p). \quad ↯$$

Case (2) $l(\gamma_s) < l(\sigma_s)$. Then σ_s is not minimizing, hence $\exists$ a cut pt $\sigma_s(\bar{t})$, $\bar{t} < l$, of p along σ_s. ↯
Case (3) $l(\gamma_s) > l(\sigma_s)$. Similarly. ↯

$\square$

Cor. Let (M^n, g) be cpl. let $p \in M$. ~~let~~ $q \in C(p)$ be such that

$$d(p,q) = i(M).$$

Suppose then q is not conjugate to p along any minimizing geodesics from p to q. Then $\exists$ a closed geodesic made up of two minimal geodesics from p to q.

Proof. Thm 4 tells $\exists$ two min. geo. γ and σ from p to q with $\gamma'(l) = -\sigma'(l)$. Our assumption implies that

$p \in C(q)$ is the closest one in $C(q)$ to q. therefore, $\gamma'(0) = -\sigma'(0)$. □

Thm5. Let (M^n, g) be ~~compact~~ ~~complete~~ compact, with

$$0 < \cancel{K_{min} \le} K \le K_{max}$$

Then a) $i(M) \ge \frac{\pi}{\sqrt{K_{max}}}$ or

b) $\exists$ a closed geodesic γ in M, whose length is less than that of any other closed geodesic in M, and which is such that

$$i(M) = \frac{1}{2} l(\gamma)$$

Proof. Let $p \in M$, $q \in C(p)$ be such that

$$d(p, q) = i(M).$$

If q is conjugate to p along some minimizing geodesic from p to q, then by Morse-Schoenberg comparison theorem,

$$d(p, q) \ge \frac{\pi}{\sqrt{K_{max}}}$$

That is, $i(M) \ge \frac{\pi}{\sqrt{K_{max}}}$.

If q is not conjugate to p along any minimizing geodesic from p to q, by ~~Theorem~~ Cor, $\exists$ a closed geodesic γ s.t.

$$i(M) = \frac{1}{2} l(\gamma).$$

γ has to have minimal length among all closed geodesics. Since otherwise, if $\exists$ a closed geodesic σ with

$$l(\sigma) < l(\gamma).$$

Let $p' \in \sigma$, and $q' \in \sigma$ s.t. $d(p', q') = \frac{1}{2} l(\sigma)$. Therefore ~~is a cut point~~ $d(p', C(p')) \le d(p', q') = \frac{1}{2} l(\sigma) < \frac{1}{2} l(\gamma)$.

Contradiction. □