微分几何(H)

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解如下方程. 找映射 $r: D \longrightarrow E^3$ 使得

$$\begin{cases} \langle r_u, r_u \rangle = E \\ \langle r_u, r_v \rangle = F \\ \langle r_v, r_v \rangle = G \end{cases} \qquad E, F, G: D \to E^3$$

$$\begin{cases} \langle r_{uu}, \frac{r_u \wedge r_v}{|r_u \wedge r_v|} \rangle = L \\ \langle r_{uv}, \frac{r_u \wedge r_v}{|r_u \wedge r_v|} \rangle = M \\ \langle r_{vv}, \frac{r_u \wedge r_v}{|r_u \wedge r_v|} \rangle = N \end{cases}$$

非线性
偏微分方程组

$$\begin{cases} \frac{\partial}{\partial u^\alpha} r = \underline{r_\alpha} \\ \frac{\partial}{\partial u^\beta} r_\alpha = \underline{\Gamma^\gamma_{\alpha\beta} r_\gamma + b_{\alpha\beta} n} \qquad \alpha, \beta = 1, 2 \\ \frac{\partial}{\partial u^\alpha} n = -\underline{b^\beta_\alpha r_\beta} \end{cases} \qquad \begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix} \in \mathbb{R}^{12}$$

一阶线性偏微分方程组

必要条件 mixed partials equal.

① $r_{\alpha\beta} = r_{\beta\alpha}$ 平凡.

② $r_{\alpha\beta\gamma} = r_{\alpha\gamma\beta}$

③ $n_{\alpha\beta} = n_{\beta\alpha}$

② $\Rightarrow$

$$\begin{cases} \frac{\partial \Gamma^\xi_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma^\xi_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\eta_{\alpha\beta}\Gamma^\xi_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma}\Gamma^\xi_{\eta\beta} = b_{\alpha\beta} b^\xi_\gamma - b_{\alpha\gamma} b^\xi_\beta \quad (G) \\ \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\xi_{\alpha\beta} b_{\xi\gamma} - \Gamma^\xi_{\alpha\gamma} b_{\xi\beta} = 0 \quad (C) \end{cases}$$

Codazzi - Mainardi 方程, 1868-1869, 1856; 俄 Karl Mikhailovich Peterson 1853

③ $n_{\alpha\beta} = n_{\beta\alpha}$

$$\Leftrightarrow \frac{\partial}{\partial u^\beta}\left(-b^\xi_\alpha r_\xi\right) = \frac{\partial}{\partial u^\alpha}\left(-b^\xi_\beta r_\xi\right)$$

$$LHS = -\frac{\partial b^\xi_\alpha}{\partial u^\beta} r_\xi - b^\xi_\alpha\left(\Gamma^\eta_{\xi\beta} r_\eta + b_{\xi\beta} n\right)$$

$$\text{LHS} = -\frac{\partial b_\alpha^\xi}{\partial u^\beta} r_\xi - b_\alpha^\xi \left(\Gamma_{\xi\beta}^\eta r_\eta + b_{\xi\beta}\, n \right)$$

$$= \underline{\left(-\frac{\partial b_\alpha^\xi}{\partial u^\beta} - b_\alpha^\eta \Gamma_{\eta\beta}^\xi \right) r_\xi - b_\alpha^\xi b_{\xi\beta}\, n} \; -$$

$$\Longleftrightarrow \begin{cases} \dfrac{\partial b_\beta^\xi}{\partial u^\alpha} - \dfrac{\partial b_\alpha^\xi}{\partial u^\beta} + b_\beta^\eta \Gamma_{\eta\alpha}^\xi - b_\alpha^\eta \Gamma_{\eta\beta}^\xi = 0 & (A) \\ b_\beta^\xi b_{\xi\alpha} - b_\alpha^\xi b_{\xi\beta} = 0 & (B) \end{cases}$$

(B) 平凡 $\quad \underline{b_{\beta\eta} g^{\eta\xi} b_{\xi\alpha}} = \underline{b_{\alpha\eta} g^{\eta\xi} b_{\xi\beta}}$

$$\| $$

$$b_{\eta\beta} g^{\xi\eta} b_{\alpha\xi}$$

$$\frac{\partial b_\beta^\xi}{\partial u^\alpha} - \frac{\partial b_\alpha^\xi}{\partial u^\beta} + b_\beta^\eta \Gamma_{\eta\alpha}^\xi - b_\alpha^\eta \Gamma_{\eta\beta}^\xi = 0 \qquad (A)$$

$$\frac{\partial b_\beta^\xi}{\partial u^\alpha} - \frac{\partial b_\alpha^\xi}{\partial u^\beta} + b_\beta^\eta \Gamma_{\eta\alpha}^\xi - b_\alpha^\eta \Gamma_{\eta\beta}^\xi = 0 \qquad (A)$$

$$\underline{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}} \quad \text{对称}$$

$\beta\alpha$ - term

$$\frac{\partial b_\beta^\xi}{\partial u^\alpha} - \frac{\partial b_\alpha^\xi}{\partial u^\beta} + b_\beta^\eta \Gamma_{\eta\alpha}^\xi - b_\alpha^\eta \Gamma_{\eta\beta}^\xi = 0$$

$$\Longleftrightarrow \frac{\partial}{\partial u^\alpha}\left(b_{\beta\gamma} g^{\gamma\xi}\right) - \frac{\partial}{\partial u^\beta}\left(b_{\alpha\gamma} g^{\gamma\xi}\right) + b_{\beta\gamma} g^{\gamma\eta} \Gamma_{\eta\alpha}^\xi - b_{\alpha\gamma} g^{\gamma\eta} \Gamma_{\eta\beta}^\xi = 0$$

$$\Longleftrightarrow \underline{\frac{\partial b_{\beta\gamma}}{\partial u^\alpha}} g^{\gamma\xi} - \underline{\frac{\partial b_{\alpha\gamma}}{\partial u^\beta}} g^{\gamma\xi} + b_{\beta\gamma} \frac{\partial}{\partial u^\alpha} g^{\gamma\xi} - b_{\alpha\gamma} \frac{\partial g^{\gamma\xi}}{\partial u^\beta}$$

$$+ \left(b_{\beta\gamma} \Gamma_{\eta\alpha}^\xi - b_{\alpha\gamma} \Gamma_{\eta\beta}^\xi \right) g^{\gamma\eta} = 0$$

两边同乘 $\underline{g_{\delta\xi}}$ 对ξ求和

$$\Longleftrightarrow \underline{g_{\delta\xi}} \left[\left(\frac{\partial b_{\beta\gamma}}{\partial u^\alpha} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} \right) \underline{g^{\gamma\xi}} + b_{\beta\gamma} \frac{\partial g^{\gamma\xi}}{\partial u^\alpha} - b_{\alpha\gamma} \frac{\partial g^{\gamma\xi}}{\partial u^\beta} \right]$$

$g_{\delta\xi} g^{\gamma\xi} = \delta_\delta^\gamma$

$$+ g_{\delta\xi} \left(b_{\beta\gamma} \Gamma_{\eta\alpha}^\xi - b_{\alpha\gamma} \Gamma_{\eta\beta}^\xi \right) g^{\gamma\eta} = 0$$

$$\Longleftrightarrow \left(\frac{\partial b_{\beta\delta}}{\partial u^\alpha} - \frac{\partial b_{\alpha\delta}}{\partial u^\beta} \right) + b_{\beta\gamma} g_{\delta\xi} \frac{\partial g^{\gamma\xi}}{\partial u^\alpha} - b_{\alpha\gamma} g_{\delta\xi} \frac{\partial g^{\gamma\xi}}{\partial u^\beta}$$

$$\Longleftrightarrow \left(\frac{\partial b_{\beta\delta}}{\partial u^\alpha} - \frac{\partial b_{\alpha\delta}}{\partial u^\beta}\right) + b_{\beta\gamma}\, g_{\delta\xi} \frac{\partial g^{\gamma\xi}}{\partial u^\alpha} - b_{\alpha\gamma}\, g_{\delta\xi}\frac{\partial g^{\gamma\xi}}{\partial u^\beta}$$
$$+ \left(b_{\beta\gamma}\Gamma_{\delta\eta\alpha} - b_{\alpha\gamma}\Gamma_{\delta\eta\beta}\right) g^{\gamma\eta} = 0$$

$$g_{\delta\xi}\frac{\partial g^{\gamma\xi}}{\partial u^\alpha} = \frac{\partial}{\partial u^\alpha}\Big(\underbrace{g_{\delta\xi}\, g^{\gamma\xi}}_{=\,\delta^\gamma_\xi \text{ 常数求导}}\Big) - g^{\gamma\xi}\frac{\partial g_{\delta\xi}}{\partial u^\alpha} = -\, g^{\gamma\xi}\, g_{\delta\xi,\alpha}$$

$$\Longleftrightarrow \left(\frac{\partial b_{\beta\delta}}{\partial u^\alpha} - \frac{\partial b_{\alpha\delta}}{\partial u^\beta}\right) \cancel{- b_{\beta\gamma}\, g^{\gamma\xi}\, g_{\delta\xi,\alpha}} \cancel{+ b_{\alpha\gamma}\, g^{\gamma\xi}\, g_{\delta\xi,\beta}}$$
$$+ b_{\beta\gamma}\, g^{\gamma\eta}\, \frac{1}{2}\left\{g_{\delta\alpha,\eta} + g_{\eta\delta,\alpha} - g_{\eta\alpha,\delta}\right\}$$
$$- b_{\alpha\gamma}\, g^{\gamma\eta}\, \frac{1}{2}\left\{g_{\delta\beta,\eta} + g_{\eta\delta,\beta} - g_{\eta\beta,\delta}\right\} = 0$$

$$\Longleftrightarrow \frac{\partial b_{\beta\delta}}{\partial u^\alpha} - \frac{\partial b_{\alpha\delta}}{\partial u^\beta} + b_{\alpha\gamma}\Gamma^\gamma_{\delta\beta} - b_{\beta\gamma}\Gamma^\gamma_{\delta\alpha} = 0$$

$$(G)\quad \frac{\partial \Gamma^\xi_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma^\xi_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\eta_{\alpha\beta}\Gamma^\xi_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma}\Gamma^\xi_{\eta\beta} = \underline{b_{\alpha\beta}\, b^\xi_\gamma - b_{\alpha\gamma}\, b^\xi_\beta}$$
$$\alpha, \beta, \gamma, \xi = 1, 2$$

$$\underline{g_{\xi\delta}\, b^\xi_\gamma} = g^{\xi\eta}\, \underline{b_{\gamma\eta}}\, g_{\xi\delta} = \underline{b_{\gamma\delta}}$$

$$g_{\delta\xi}\left(\frac{\partial \Gamma^\xi_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma^\xi_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\eta_{\alpha\beta}\Gamma^\xi_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma}\Gamma^\xi_{\eta\beta}\right) = b_{\alpha\beta}\, b_{\gamma\delta} - b_{\alpha\gamma}\, b_{\beta\delta}$$

Riemann 记号

$$R_{\delta\alpha\beta\gamma} = -\, g_{\delta\xi}\left(\frac{\partial \Gamma^\xi_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma^\xi_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\eta_{\alpha\beta}\Gamma^\xi_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma}\Gamma^\xi_{\eta\beta}\right)$$

$$(G) \Longleftrightarrow \quad \underline{R_{\delta\alpha\beta\gamma}} = b_{\alpha\beta}\, b_{\gamma\delta} - b_{\alpha\gamma}\, b_{\beta\delta} \qquad \delta, \alpha, \beta, \gamma = 1, 2$$

对称性 $\quad R_{\delta\alpha\beta\gamma} = -R_{\delta\alpha\gamma\beta} = -R_{\alpha\delta\beta\gamma} = R_{\beta\gamma\delta\alpha}$

Claim $\quad R_{\delta\alpha\beta\gamma} = \frac{1}{2}\left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^\gamma \partial u^\alpha} + \frac{\partial^2 g_{\alpha\beta}}{\partial u^\gamma \partial u^\delta} + \frac{\partial^2 g_{\delta\gamma}}{\partial u^\beta \partial u^\alpha} - \frac{\partial^2 g_{\alpha\gamma}}{\partial u^\beta \partial u^\delta}\right)$

作业 $\quad + g^{\eta\xi}\left(\Gamma_{\xi\alpha\beta}\Gamma_{\eta\delta\gamma} - \Gamma_{\xi\alpha\gamma}\Gamma_{\eta\delta\beta}\right)$

$$\boxed{R_{1212}} = b_{11}\, b_{22} - b_{12}^2 = \det\begin{pmatrix} L & M \\ M & N\end{pmatrix}$$

$$\boxed{R_{1212}} \Big/ \det\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \frac{b_{11}b_{22}-b_{12}^2}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} = \frac{\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} = K \quad \text{Gauss cur.}$$

$$K = \boxed{\frac{R_{1212}}{g_{11}g_{22}-g_{12}^2}}$$ 由 $\{g_{\alpha\beta}\}$ 确定 高斯绝妙定理

Codazzi 方程

$$\frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} + \Gamma_{\alpha\beta}^{\xi} b_{\xi\gamma} - \Gamma_{\alpha\gamma}^{\xi} b_{\xi\beta} = 0 \qquad \alpha,\beta,\gamma = 1,2$$

关 β,γ 反称

$$\begin{cases} \frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} + \Gamma_{11}^{\xi} b_{\xi 2} - \Gamma_{12}^{\xi} b_{\xi 1} = 0 \\ \frac{\partial b_{21}}{\partial u^2} - \frac{\partial b_{22}}{\partial u^1} + \Gamma_{21}^{\xi} b_{\xi 2} - \Gamma_{22}^{\xi} b_{\xi 1} = 0 \end{cases}$$

Gauss 方程, Codazzi 方程 <u>曲面论结构方程</u>

<u>Claim</u> 假设 $r(u,v)$ 参数化 满 $M=F=0$, 上述两方程等价于

$$\begin{cases} L_v = H E_v \\ N_u = H G_u \end{cases}$$

<u>曲面论基本定理</u>

$$\begin{cases} \frac{\partial}{\partial u^1}\begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}\begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix} \\ \frac{\partial}{\partial u^2}\begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}\begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix} \end{cases} \qquad \begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix}(u_0,v_0) = \begin{pmatrix} r^0 \\ r_1^0 \\ r_2^0 \\ n^0 \end{pmatrix}$$

<u>Thm</u> (Spivak I, pp. 187) $F: D^{\subseteq \mathbb{R}^2} \longrightarrow V \subseteq \mathbb{R}^n$, $x \in V$

$$\begin{cases} F(0) = x & 0 \in D \\ \frac{\partial F}{\partial u}(u^1,u^2) = f(u^1,u^2, F(u^1,u^2)) \end{cases}$$

$$\left\{ \frac{\partial F}{\partial u^\alpha}(u^1,u^2) = f_\alpha(\underline{u^1,u^2}_{D},\ \underline{F(u^1,u^2)}_{V}) \right.$$

where $f_\alpha : D\times V \longrightarrow \mathbb{R}^n$ C^∞ 函数.

① 唯一性 若有两个解 $F_1 : D_1 \to V$，$F_2 : D_2 \to V$ 则 $F_1 = F_2$ on $D_1\cap D_2$

② 存在性 若 $\frac{\partial}{\partial u^\beta} f_\alpha(u^1,u^2,F(u^1,u^2)) = \frac{\partial}{\partial u^\alpha} f_\beta(u^1,u^2,F(u^1,u^2))$

则存在 $0\in D$ 的一个小邻域 $\underline{\underline{W}}$，和 C^∞ $F: W\to V$ 满足方程

如果方程是线性的，则可取 $W=D$.

定理1 (唯一性) 设 M_1, M_2 是定义在同一个区域 D 上的两个正则曲面，它们参数表示为 $r(u^1,u^2)$，$\tilde{r}(u^1,u^2)$

如果，$\forall (u^1,u^2)\in D$，M_1, M_2 在点 $r(u^1,u^2)$ 处的 I 和 II 相同 (i.e. E,F,G,L,M,N 作为 D 上函数相同)

则 M_1, M_2 至多相差一个 E^3 的刚体运动

Pf. □

定理 (存在性) 给定 $E,F,G,L,M,N: D\to E^3$ 要求 $\begin{pmatrix} E & F \\ F & G\end{pmatrix} > 0$

(可进一步得到 $g_{\alpha\beta}$, $\Gamma^\gamma_{\alpha\beta}$, $b_{\alpha\beta}$, b^β_α) 满足 (G) 和 (C)

则存在一个正则曲面片 $r: \underline{D} \to E^3$ 使

$$\langle r_u, r_u\rangle = E \qquad \langle r_{uu}, n\rangle = L$$
$$\langle r_u, r_v\rangle = F \qquad \langle r_{uv}, n\rangle = M$$
$$\langle r_v, r_v\rangle = G \qquad \langle r_{vv}, n\rangle = N$$

证明. 解方程

$$\begin{cases} \frac{\partial r}{\partial u^\alpha} = r_\alpha \\ \frac{\partial r_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta} r_\gamma + b_{\alpha\beta} n \\ \frac{\partial n}{\partial u^\alpha} = -b^\beta_\alpha r_\beta \end{cases}$$

在 (G), (C) 下，有解 (r)

在(G),(C)下，存唯 给定初值 $\begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix} : D \to E^{12}$

存 $r: D \to E^3$

给初值 r^0, r^0_α, n^0 s.t. $\begin{cases} \langle r^0_\alpha, r^0_\beta \rangle = g_{\alpha\beta}(u_0^1, u_0^2) \\ \langle r^0_\alpha, n^0 \rangle = 0 \\ \langle n^0, n^0 \rangle = 1 \\ (r^0_1, r^0_2, n) > 0 \end{cases}$

$$\begin{cases} \dfrac{\partial \langle r_\alpha, r_\beta \rangle}{\partial u^\gamma} = \langle r_{\alpha\gamma}, r_\beta \rangle + \langle r_\alpha, r_{\beta\gamma} \rangle \\ \qquad = \Gamma^\xi_{\alpha\gamma} \langle r_\xi, r_\beta \rangle + b_{\alpha\gamma} \langle n, r_\beta \rangle + \Gamma^\xi_{\beta\gamma} \langle r_\alpha, r_\xi \rangle + b_{\beta\gamma} \langle r_\alpha, n \rangle \\ \dfrac{\partial \langle r_\alpha, n \rangle}{\partial u^\gamma} = \\ \dfrac{\partial \langle n, n \rangle}{\partial u^\alpha} = \end{cases}$$

解存在唯一性　检查

$\begin{cases} \langle r_\alpha, r_\beta \rangle = g_{\alpha\beta} \\ \langle r_\alpha, n \rangle = 0 \\ \langle n, n \rangle = 1 \end{cases}$ 是上述方程的解

唯一性 $\Rightarrow$ 存在 $\langle r_\alpha, r_\beta \rangle = g_{\alpha\beta}$ $\leftarrow$

$\langle r_\alpha, n \rangle = 0$

$\langle n, n \rangle = 1$

$r: D \to E^3$

$(r^0_1, r^0_2, n^0) > 0$

(r_1, r_2, n) 连续 in

$\Rightarrow (r_1, r_2, n) > 0$

$\Rightarrow$ 找到曲面 $r: D \to E^3$

s.t. $\begin{cases} \langle r_u, r_u \rangle = E \quad \langle r_u, r_v \rangle = F, \ \langle r_v, r_v \rangle = G \\ \langle r_{uu}, n \rangle = L, \ \langle r_{uv}, n \rangle = M, \ \langle r_{vv}, n \rangle = N. \end{cases}$ □

可积性条件 integrability condition

□