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正交活动标架

$r = r(u,v)$ $\quad \{r_u, r_v, n\} = \{r_1, r_2, n\}$ 自然标架

曲线 $c = c(s)$, $\{t(s), n(s), b(s)\}$ Frenet 标架

曲面

$$\begin{cases} \dfrac{\partial r}{\partial u^\alpha} = r_\alpha \\ \dfrac{\partial r_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta} r_\gamma + b_{\alpha\beta} n \\ \dfrac{\partial n}{\partial u^\alpha} = -b^\gamma_\alpha r_\gamma \end{cases}$$

曲线

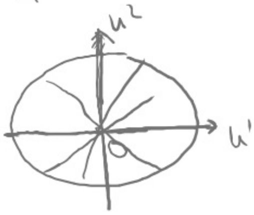
$$\begin{cases} \dfrac{dc}{ds} = t(s) \\ \dfrac{dt}{ds} = \underline{k(s)} n(s) \\ \dfrac{dn}{ds} = -k(s)t(s) + \tau(s) b(s) \\ \dfrac{db}{ds} = -\tau(s) n(s) \end{cases}$$

$$\frac{\partial}{\partial u^1}\begin{pmatrix} r_1 \\ r_2 \\ n \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ n \end{pmatrix}$$

$$\frac{d}{ds}\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

反称

$\dfrac{\partial}{\partial u^2}$



$r, r_1, r_2, n : \underset{\subset \mathbb{R}^2}{D} \to E^3$

$V = a\dfrac{\partial}{\partial u^1} + b\dfrac{\partial}{\partial u^2}$

$c'(t) = V$

$c(t) = (u^1(t), u^2(t))$

找曲线 $c(t)$ 使 $c(0) = 0$, $c'(0) = V$

$$\left.\frac{d}{dt}\right|_{t=0} r(c(t)) = \left.\frac{d}{dt}\right|_{t=0} r(u^1(t), u^2(t)) = r_1\frac{du^1}{dt} + r_2\frac{du^2}{dt} = a r_1 + b r_2$$

$$\begin{cases} dr = (du^\alpha) r_\alpha & \alpha = 1,2 \\ dr_\alpha = \left(\Gamma^\gamma_{\alpha\beta} du^\beta\right) r_\gamma + \left(b_{\alpha\beta} du^\beta\right) n & \alpha = 1,2 \\ dn = \left(-b^\gamma_\alpha du^\alpha\right) r_\gamma \end{cases}$$

$r : D \to E^3$ $\quad x, y, z : D \to \mathbb{R}$

$r(u,v) = (x(u,v), y(u,v), z(u,v))$

$$dn = \frac{\partial n}{\partial u^\alpha} du^\alpha = -b^\gamma_\alpha r_\gamma du^\alpha$$

$$dr = (dx, dy, dz) = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv, \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right)$$

$$\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv)$$

$$= \underbrace{\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)}du + \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)dv$$

$$= r_u \underline{du} + r_v \underline{dv}$$

$$dr_\alpha = \frac{\partial r_\alpha}{\partial u}du + \frac{\partial r_\alpha}{\partial v}dv = \frac{\partial r_\alpha}{\partial u^\beta}du^\beta = \left(\Gamma_{\alpha\beta}^\gamma r_\gamma + b_{\alpha\beta}n\right)du^\beta$$

$$= \underline{\left(\Gamma_{\alpha\beta}^\gamma du^\beta\right)} r_\gamma + \underline{\left(b_{\alpha\beta}du^\beta\right)} n$$

$$dr = (du^1) r_1 + (du^2) r_2$$

$$\begin{cases} dr_1 = \left(\Gamma_{1\beta}^1 du^\beta\right) r_1 + \left(\Gamma_{1\beta}^2 du^\beta\right) r_2 + \left(b_{1\beta}du^\beta\right) n \\ dr_2 = \left(\Gamma_{2\beta}^1 du^\beta\right) r_1 + \left(\Gamma_{2\beta}^2 du^\beta\right) r_2 + \left(b_{2\beta}du^\beta\right) n \\ dn = \left(-b_\alpha^1 du^\alpha\right) r_1 + \left(-b_\alpha^2 du^\alpha\right) r_2 \end{cases}$$

$$d\begin{pmatrix} r_1 \\ r_2 \\ n \end{pmatrix} = \underline{\begin{pmatrix} \Gamma_{1\beta}^1 du^\beta & \Gamma_{1\beta}^2 du^\beta & b_{1\beta}du^\beta \\ \Gamma_{2\beta}^1 du^\beta & \Gamma_{2\beta}^2 du^\beta & b_{2\beta}du^\beta \\ -b_\alpha^1 du^\alpha & -b_\alpha^2 du^\alpha & 0 \end{pmatrix}} \begin{pmatrix} r_1 \\ r_2 \\ n \end{pmatrix}$$

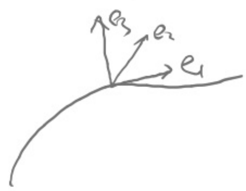
"正交性" ⇒ 反称.

例子. $c(s)$ 是 E^3 中一条弧长参数的正则曲线

每点处取一个正定向的单位正交标架 $\{c;\ e_1(s), e_2(s), e_3(s)\}$

光滑的依赖于 s

$$\frac{dc}{ds} = a^1 e_1 + a^2 e_2 + a^3 e_3$$

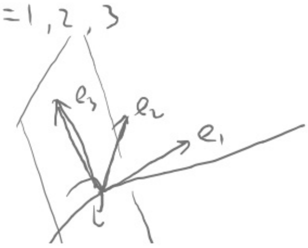


取 $e_1(s) = \frac{dc}{ds}$ 即 $a^1 = 1$, $a^2 = 0$, $a^3 = 0$

标架的运动方程

$$\frac{d e_i(s)}{ds} = q_i^j(s)\, e_j(s), \quad i = 1, 2, 3$$

$$\frac{d}{ds}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & q_1^2 & q_1^3 \\ q_2^1 & 0 & q_2^3 \\ q_3^1 & q_3^2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$



$$\frac{d}{ds}\begin{pmatrix} e_2 \\ e_3 \end{pmatrix} = \underbrace{\begin{pmatrix} q_2^1 & 0 & q_2^3 \\ q_3^1 & q_3^2 & 0 \end{pmatrix}}\begin{pmatrix} e_2 \\ e_3 \end{pmatrix}$$

$\langle e_i, e_j\rangle(s) \equiv \delta_{ij}$

求导 $0 = \frac{d}{ds}\langle e_i, e_j\rangle = \langle \frac{de_i}{ds}, e_j\rangle + \langle e_i, \frac{de_j}{ds}\rangle$

$$= \langle q_i^k e_k, e_j\rangle + \langle e_i, q_j^l e_l\rangle$$

$$= q_i^k \delta_{kj} + q_j^l \delta_{il}$$

$$= q_i^j + q_j^i$$

“斜对称” $\Rightarrow$ $0 = q_i^j + q_j^i$ i.e. $q_i^j = -q_j^i$

重新选取标架 $\{c, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$

$$\begin{cases} \tilde{e}_1 = e_1 \\ \begin{pmatrix} \tilde{e}_2 \\ \tilde{e}_3 \end{pmatrix}(s) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e_2 \\ e_3 \end{pmatrix} \end{cases}$$

$$\frac{d\tilde{e}_i}{ds} = \tilde{q}_i^j \tilde{e}_j$$

$$\frac{d\tilde{e}_1}{ds} = \frac{de_1}{ds} = \underline{q_1^2 e_2 + q_1^3 e_3}$$

$$\tilde{q}_1^2 \tilde{e}_2 + \tilde{q}_1^3 \tilde{e}_3 = \tilde{q}_1^2(\cos\theta\, e_2 + \sin\theta\, e_3) + \tilde{q}_1^3(-\sin\theta\, e_2 + \cos\theta\, e_3)$$

$$= \underline{\left[\tilde{q}_1^2 \cos\theta + \tilde{q}_1^3(-\sin\theta)\right] e_2 + \left(\tilde{q}_1^2 \sin\theta + \tilde{q}_1^3 \cos\theta\right) e_3}$$

$$\Longleftrightarrow \quad \begin{pmatrix} q_1^2 \\ q_1^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \tilde{q}_1^2 \\ \tilde{q}_1^3 \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} \tilde{q}_1^2 \\ \tilde{q}_1^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} q_1^2 \\ q_1^3 \end{pmatrix}$$

$\left|\frac{de_1}{ds}\right| > 0$ $\quad (q_1^2)^2 + (q_1^3)^2 > 0$. 不妨设 $q_1^2 \neq 0$ $\quad K(s) = \sqrt{(q_1^2)^2 + (q_1^3)^2}$

$\left|\frac{d\alpha}{ds}\right| > 0 \quad (q_1^2)^2 + (q_1^3)^2 > 0.$ 不妨设 $q_1^2 \neq 0 \quad K(s) = \sqrt{(q_1^2)^2 + (q_1^3)^2}$

$$\tilde{q}_1^3 = -\sin\theta\, q_1^2 + \cos\theta\, q_1^3$$

取 θ. 使得 $\boxed{\tan\theta = \frac{q_1^3}{q_1^2}}$

$$\tilde{q}_1^2 = \cos\theta\, q_1^2 + \sin\theta\, q_1^3$$

Claim. 总能选 $\theta(s)$ 使得 $\tilde{q}_1^3 \equiv 0$, $\boxed{\tilde{q}_1^2 > 0}$ (思考题)

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$$\frac{d}{ds}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix} = \begin{pmatrix}0 & \tilde{q}_1^2 & 0\\ -\tilde{q}_1^2 & 0 & \tilde{q}_2^3\\ 0 & -\tilde{q}_2^3 & 0\end{pmatrix}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix} \qquad \frac{d\tilde{e}_1}{ds} = \underbrace{\boxed{\tilde{q}_1^2}}_{>0}\,\overbrace{\boxed{\tilde{e}_2}}^{n}$$

$r = r(u,v) \qquad \{r_u, r_v, n\}$

假设 标准正交. $\langle r_\alpha, r_\beta\rangle = \delta_{\alpha\beta} \Rightarrow g_{\alpha\beta} = \delta_{\alpha\beta}$

$\Rightarrow K = 0$

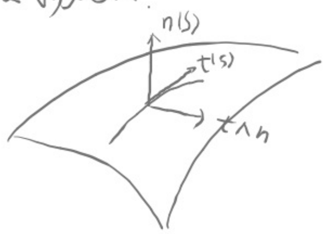
$\{e_1(u,v),\ e_2(u,v),\ \overset{n(u,v)}{\overset{\|}{e_3}}(u,v)\}$

曲面上的正交标架

"活动标架" Moving frames 法国数学家 Cotton, Darboux.

Frenet, Serret Élie Cartan 发扬光大.

Darboux 曲面上的曲线的活动标架



定义(光滑向量场)

正则曲面片 $r = r(u,v)$ 上的光滑向量场 $X(u,v)$ 是指对曲面上任一点 P 给定一个该点 $\overset{r(u_0,v_0)}{\overset{\|}{P}}$ 处的向量 $X(u_0,v_0)$, 且 $X(u,v)$ 光滑地依赖于 (u,v)

定义(活动标架场) $r = r(u,v)$ 上的任一点处给定 E^3 的标架

$$\{r(u,v);\ X_1(u,v),\ X_2(u,v),\ X_3(u,v)\}$$

其中要求 X_i 光滑向量场

一般地, 我们要求 $(X_1, X_2, X_3) > 0$

一般地，我们要求 $(X_1, X_2, X_3) > 0$

如果 $\{X_1, X_2, X_3\}$ 是单位正交标架，则称 $\{r(u,v); X_1, X_2, X_3\}$ 为曲面的正交活动标架

是否总存在？ 从 $\{r_u, r_v, n\}$ 出发

Schmit 正交化.

$$\begin{cases} e_1 = \dfrac{r_u}{|r_u|} \leftarrow \\ e_2 = \dfrac{r_v - \langle r_v, e_1\rangle e_1}{|r_v - \langle r_v, e_1\rangle e_1|} \leftarrow \\ \boxed{e_3 = n} = e_1 \wedge e_2 \end{cases}$$

$$\begin{cases} dr = r_1 du^1 + r_2 du^2 \\ dr_\alpha = (\Gamma_{\alpha\beta}^\gamma du^\beta) r_\gamma + b_{\alpha\beta} du^\beta n \\ dn = (-b_\alpha^\gamma du^\alpha) r_\gamma \end{cases}$$

$r : D \to E^3$

(u^1, u^2) D 上的坐标函数

$du^\alpha \left(\dfrac{\partial}{\partial u^\beta}\right) = \delta_\beta^\alpha$

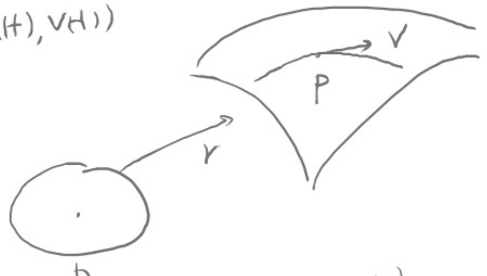
对任意 D 的切向量 $V \in T_pD$

$$\boxed{dr(V) = r_1 du^1(V) + r_2 du^2(V) \in T_{r(p)} r(D)}$$

$$\begin{cases} \dfrac{\partial}{\partial u^1} \longmapsto r_1 \\ \dfrac{\partial}{\partial u^2} \longmapsto r_2 \end{cases}$$

$dr_\alpha(V)$ $\quad C(t) = r(u(t), v(t))$

$= \dfrac{d}{dt}\Big|_{t=0} r_\alpha(C(t))$



u, v : 曲面 $\to \mathbb{R}$ $\quad u(C(t)), \ v(C(t))$

$du(r_u) = \dfrac{d}{du}\Big|_{u=u_0} u(u \mapsto r(u, v_0)) = 1$ $\quad dv(r_u) = 0$

$du(r_v) = \dfrac{d}{dv}\Big|_{v=v_0} u(v \mapsto r(u_0, v)) = 0$ $\quad dv(r_v) = 1$

正交活动标架的运动方程

$\{r(u,v); e_1, e_2, e_3\}$, $e_3 = n$

$\{r_1, r_2\}$, $\{e_1, e_2\}$ 是相应点处切平面的一组基

$dr = r_\alpha du^\alpha$

$\begin{pmatrix} r_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 \end{pmatrix} \begin{pmatrix} e_1 \\ \end{pmatrix}$

$\overset{\parallel}{n}$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

i.e. $\boxed{r_\alpha = a_\alpha^\beta e_\beta}$

$$\begin{aligned} dr &= r_\alpha du^\alpha \\ &= a_\alpha^\beta e_\beta du^\alpha \\ &= (a_\alpha^\beta du^\alpha) e_\beta \\ &= (a_\alpha^1 du^\alpha) e_1 + (a_\alpha^2 du^\alpha) e_2 \\ &= (a_1^1 du^1 + a_2^1 du^2) e_1 + (a_1^2 du^1 + a_2^2 du^2) e_2 \\ &=: \omega^1 e_1 + \omega^2 e_2 \end{aligned}$$

$du^\alpha(p): T_pM \to \mathbb{R}$

定义 1-形式 $\omega^\alpha = a_\beta^\alpha du^\beta$, $\alpha = 1, 2$

$dr = \omega^1 e_1 + \omega^2 e_2$

ω^α 给定一个切向量场 $X = X^1 r_1 + X^2 r_2 = X^\alpha r_\alpha$

$$\begin{aligned} \omega^\alpha(X) &= \omega^\alpha(X^\eta r_\eta) = X^\eta a_\beta^\alpha du^\beta(r_\eta) = X^\eta a_\beta^\alpha \delta_\eta^\beta \\ &= X^\eta a_\eta^\alpha \end{aligned}$$

$$\begin{aligned} \langle X, e_\alpha \rangle = \langle X^\eta r_\eta, e_\alpha \rangle &= X^\eta \langle r_\eta, e_\alpha \rangle \\ &= X^\eta \langle a_\eta^\beta e_\beta, e_\alpha \rangle \\ &= X^\eta a_\eta^\beta \delta_{\beta\alpha} = X^\eta a_\eta^\alpha \end{aligned}$$

$\Rightarrow \quad \omega^\alpha(X) = \langle X, e_\alpha \rangle$, $\alpha = 1, 2$

ω^α 是 e_α 的对偶 1-形式

$dr = \omega^1 e_1 + \omega^2 e_2$

$X \quad dr(X) = \omega^1(X) e_1 + \omega^2(X) e_2$ $\quad \omega^1, \omega^2$

$$\begin{aligned} &= \langle X, e_1 \rangle e_1 + \langle X, e_2 \rangle e_2 \\ &= X \end{aligned}$$

dr_α, dn 下课