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记号: $\alpha,\beta=1,2$

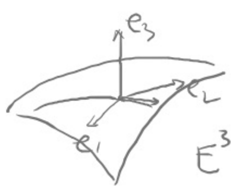
$i,j,k=1,2,3$ $\quad r=r(u,v)$

正交活动标架

$$\{r;\ e_1,\ e_2,\ \underset{n}{\overset{\|}{e_3}}\}$$

$e_i(u,v)$

$\langle e_i, e_j\rangle \equiv \delta_{ij}$



$$\begin{cases} dr = \underline{w^1 e_1 + w^2 e_2} \\ de_1 = \\ de_2 = \\ de_3 = \end{cases}$$

切向量场

$w^\alpha(X) = \langle X, e_\alpha\rangle$, 特别地 $w^\alpha(e_\beta)=\delta^\alpha_\beta$

设 $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, $\quad \underline{r_\alpha = a_\alpha^\beta e_\beta}$

$\Longrightarrow \boxed{w^\alpha = a^\alpha_\beta du^\beta}$ $\qquad \underline{du^\beta(r_\alpha)=\delta^\beta_\alpha}$

u^β 看作曲面上点的坐标函数

$\underline{r: D \to r(D)\subset E^3}$ $\qquad r(u(p), v(p))$, $p\in$曲面片 $=p$.

对任意切向量场 X, $(w^1e_1+w^2e_2)(X) = w^1(X)e_1 + w^2(X)e_2$

$= \langle X,e_1\rangle e_1 + \langle X,e_2\rangle e_2$

$= X$

1形式 $\quad w^1,\ w^2,\ du^1,\ du^2$

曲面上之函数

du^α 在每点处是 $T_pM\to\mathbb{R}$ 的线性映射. $du^\alpha(r_\beta)=\delta^\alpha_\beta$

定义: 考虑一个正则曲面片 M. M上的1-形式 ϕ 定义为 M 上所有切向量集合上的函数. 限制在每一点 $p\in M$ 处.

$$\phi: T_pM \longrightarrow \mathbb{R}$$

是线性函数.

若对 M 上任一光滑切向量场 X, $\phi(X)$ 是光滑函数, 则称1形式 ϕ 光滑

性质: 正则曲面片 M 设 V_1, V_2 为 M 上 C^∞ 切向量场, 且逐点处 V_1, V_2 线性无关 设 θ^1, θ^2 为 M 上 C^∞ 1形式 使得

$$\boxed{\theta^\alpha(V_\beta)=\delta^\alpha_\beta}$$

那么曲面片 M 上任一 C^∞ 1-形式 ϕ 均可写成

$$\phi=\sum_{\alpha=1}^{2}\phi(V_\alpha)\theta^\alpha$$

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证明：ϕ, $\sum_{\alpha=1}^{2}\phi(V_\alpha)\theta^\alpha$ 均为光滑1-形式

对任 $p\in M$，$\forall X\in T_pM$，$\phi(X)$ 和 $\sum_{\alpha=1}^{2}\phi(V_\alpha)\theta^\alpha(X)$

$X=X^\alpha V_\alpha \quad \phi(X)=\phi(X^\alpha V_\alpha)=X^\alpha\phi(V_\alpha)$

$\phi(V_\alpha)\theta^\alpha(X)=\phi(V_\alpha)\theta^\alpha(X^\beta V_\beta)=\phi(V_\alpha)X^\alpha$ □

$\underline{\omega^1 \quad \omega^2}$ $\quad$ $\underbrace{du^1 \quad du^2}$

$\uparrow$ $\quad r_1, r_2$ $\quad du^\alpha(r_\beta)=\delta^\alpha_\beta \Rightarrow \phi=\phi(r_\alpha)du^\alpha$

$e_1 \quad e_2$

$\omega^\alpha(e_\beta)=\delta^\alpha_\beta \Rightarrow \phi=\phi(e_\alpha)\omega^\alpha$

$$\underline{de_i=\omega_i^j e_j=\omega_i^1 e_1+\omega_i^2 e_2+\omega_i^3 e_3}, \quad i=1,2,3.$$

曲面上二1-形式

$$d\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}\omega_1^1 & \boxed{\omega_1^2} & \boxed{\omega_1^3}\\ \omega_2^1 & \omega_2^2 & \boxed{\omega_2^3}\\ \omega_3^1 & \omega_3^2 & \omega_3^3\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$$

对 $p\in M$，$\forall V\in T_pM$，

在p处：$de_i(V)=\underbrace{\omega_i^j(V)}e_j$

$$d\begin{pmatrix}r_1\\r_2\\n\end{pmatrix}=\underbrace{\begin{pmatrix}\times & \times & \times\\ \times & \times & \times\\ \times & \times & \times\end{pmatrix}}\begin{pmatrix}r_1\\r_2\\n\end{pmatrix}$$

标准

正交性 $\langle e_i, e_j\rangle\equiv\delta_{ij}, \ i,j=1,2,3$

$$\begin{aligned}0=d\langle e_i,e_j\rangle&=\langle de_i,e_j\rangle+\langle e_i,de_j\rangle\\&=\langle\omega_i^k e_k,e_j\rangle+\langle e_i,\omega_j^k e_k\rangle\\&=\omega_i^j+\omega_j^i\end{aligned}$$

$\Longrightarrow \quad \omega_i^j=-\omega_j^i$ 反称

$$\begin{cases}dr=\omega^1e_1+\omega^2e_2\\ de_j=\omega_j^i e_i, \quad j=1,2,3\\ \quad\omega_j^i=-\omega_i^j\end{cases}$$

归结为五个1-形式 $\underline{\omega^1}$, $\underline{\omega^2}$

$\underline{\omega_1^2,\ \boxed{\omega_1^3,\ \omega_2^3}}$

?

第一基本形式，第二基本形式

$\mathrm{I}=\boxed{g_{\alpha\beta}}\,\underline{du^\alpha}\otimes\underline{du^\beta} \qquad \mathrm{II}=b_{\alpha\beta}\,\underline{du^\alpha}\otimes\underline{du^\beta}$

$$\mathrm{I} = \boxed{g_{\alpha\beta}}\, \underline{du^\alpha} \otimes \underline{du^\beta} \qquad \mathrm{II} = b_{\alpha\beta}\, \underline{du^\alpha} \otimes \underline{du^\beta}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

设 V, W 为两个切向量场

$$\underline{\mathrm{I}(V,W)} = \langle V, W\rangle = \big\langle \underline{\langle V,e_1\rangle} e_1 + \underline{\langle V,e_2\rangle} e_2,\ \underline{\langle W,e_1\rangle} e_1 + \underline{\langle W,e_2\rangle} e_2 \big\rangle$$

$$= \langle V,e_1\rangle\langle W,e_1\rangle + \langle V,e_2\rangle\langle W,e_2\rangle$$

$$= \underline{\omega^1(V)\,\omega^1(W) + \omega^2(V)\,\omega^2(W)}, \quad \forall\ V, W$$

$$\Longrightarrow \boxed{\mathrm{I} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2}$$

$\mathrm{II}(V,W) = \langle \mathcal{W}(V), W\rangle = \langle V, \mathcal{W}(W)\rangle$, $\mathcal{W}$: Weingarten

回忆 在 $\{r_1, r_2\}$ 基下 $\mathcal{W}\begin{pmatrix} r_1 \\ r_2\end{pmatrix} = \begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}\begin{pmatrix} r_1 \\ r_2\end{pmatrix}$ [illegible]

在 $\{e_1, e_2\}$ 基下 设 $\mathcal{W}\begin{pmatrix} e_1 \\ e_2\end{pmatrix} = \begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2\end{pmatrix}\begin{pmatrix} e_1 \\ e_2\end{pmatrix}$

i.e. $\underline{\mathcal{W}(e_\alpha) = h_\alpha^\beta e_\beta}$, $\alpha = 1, 2$

回忆. $\underline{r_\alpha = a_\alpha^\beta e_\beta} \Longrightarrow \underline{\omega^\alpha = a_\beta^\alpha du^\beta}$

$\underline{\mathcal{W}(r_\alpha) = -n_\alpha}$

自然标架

$$de_3 = dn = \underline{\underline{n_\alpha}}\, du^\alpha = -\mathcal{W}(r_\alpha)\, du^\alpha = -a_\alpha^\beta\, \mathcal{W}(e_\beta)\, du^\alpha$$

$$= -(a_\alpha^\beta du^\alpha)\, \mathcal{W}(e_\beta)$$

$$= -\underline{\omega^\beta}\, \underline{\mathcal{W}(e_\beta)}$$

$$= -\omega^\beta h_\beta^\alpha e_\alpha$$

正交活动标架

$$\| \qquad \omega_3^j e_j = \underline{\omega_3^1} e_1 + \underline{\omega_3^2} e_2$$

$$\Longrightarrow \quad -\omega^\beta h_\beta^\alpha e_\alpha = \omega_3^\alpha e_\alpha$$

$$\Longrightarrow \quad \boxed{\omega_3^\alpha = -h_\beta^\alpha \omega^\beta} \quad \alpha = 1, 2$$

$$\boxed{\begin{aligned} \omega_1^3 &= -\omega_3^1 = (h_1^1 \omega^1 + h_2^1 \omega^2) \\ \omega_2^3 &= -\omega_3^2 = h_1^2 \omega^1 + h_2^2 \omega^2 \end{aligned}} \Longrightarrow \begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

$$\langle V, \mathcal{W}(W)\rangle = \underbrace{\langle \mathcal{W}(V), W\rangle}_{\forall\ V, W}$$

$$\Longrightarrow \boxed{h_\alpha^\beta = h_\beta^\alpha}$$

$$\forall\ v, w$$

$$\mathrm{II}(v,w) = \langle v, \mathcal{W}(w) \rangle \qquad \Rightarrow \boxed{h_\alpha^\beta = h_\beta^\alpha}$$

$$= \underline{\begin{pmatrix} \langle v, e_1\rangle & \langle v, e_2\rangle \end{pmatrix}} \underline{\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix}} \begin{pmatrix} \langle w, e_1\rangle \\ \langle w, e_2\rangle \end{pmatrix}$$

$$\left(\begin{aligned} &= \langle \mathcal{W}(v), w \rangle \\ &= \begin{pmatrix} \langle v, e_1\rangle & \langle v, e_2\rangle \end{pmatrix} \underline{\begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}} \begin{pmatrix} \langle w, e_1\rangle \\ \langle w, e_2\rangle \end{pmatrix} \end{aligned} \right)$$

$$= \begin{pmatrix} \omega^1(v) & \omega^2(v) \end{pmatrix} \underline{\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} \begin{pmatrix} \omega^1(w) \\ \omega^2(w) \end{pmatrix}}$$

$$= \begin{pmatrix} \omega^1(v), \omega^2(v) \end{pmatrix} \begin{pmatrix} \omega_1^3(w) \\ \omega_2^3(w) \end{pmatrix}$$

$$= \omega^1(v)\,\omega_1^3(w) + \omega^2(v)\,\omega_2^3(w)$$

$$\Rightarrow \boxed{\mathrm{II} = \omega^1 \otimes \omega_1^3 + \omega^2 \otimes \omega_2^3}$$

性质：曲面的第一基本形式与正交活动标架的选取无关

曲面的第二基本形式与同法向的正交活动标架选取无关

证明：$\{r, e_1, e_2, \underset{n}{\overset{\shortparallel}{e_3}}\}$ $\quad \{r; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$

考虑：$e_3 = \bar{e}_3$

$$\begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} = R_\theta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \qquad R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

$$\mathrm{I} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 \qquad \bar{\mathrm{I}} = \bar{\omega}^1 \otimes \bar{\omega}^1 + \bar{\omega}^2 \otimes \bar{\omega}^2$$

$$dr = \omega^1 e_1 + \omega^2 e_2 = \bar{\omega}^1 \bar{e}_1 + \bar{\omega}^2 \bar{e}_2 = \begin{pmatrix} \bar{\omega}^1 & \bar{\omega}^2 \end{pmatrix} \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\omega}^1 & \bar{\omega}^2 \end{pmatrix} R_\theta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \omega^1 & \omega^2 \end{pmatrix} = \begin{pmatrix} \bar{\omega}^1 & \bar{\omega}^2 \end{pmatrix} R_\theta$$

$$\Rightarrow \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = R_\theta^T \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{pmatrix} \Rightarrow \underline{\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{pmatrix} = \boxed{R_\theta} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}}$$

$$\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

$$\begin{pmatrix} \overline{\omega^1} \\ \overline{\omega^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

$$\begin{aligned} \overline{\omega^1}\otimes\overline{\omega^1} + \overline{\omega^2}\otimes\overline{\omega^2} &= (\cos\theta\,\omega^1+\sin\theta\,\omega^2)\otimes(\cos\theta\,\omega^1+\sin\theta\,\omega^2) \\ &\quad + (-\sin\theta\,\omega^1+\cos\theta\,\omega^2)\otimes(-\sin\theta\,\omega^1+\cos\theta\,\omega^2) \\ &= \omega^1\otimes\omega^1+\omega^2\otimes\omega^2 \end{aligned}$$

(还需考虑 $\begin{pmatrix} \bar e_1 \\ \bar e_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}}_{\det = -1} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$)

~~故~~ $e_3 = \overline{e_3}$

$de_3 = d\overline{e_3} \iff \omega_3^1 e_1 + \omega_3^2 e_2 = \overline{\omega_3^1}\,\overline{e_1} + \overline{\omega_3^2}\,\overline{e_2}$

$$\Rightarrow \begin{pmatrix} \overline{\omega_1^3} \\ \overline{\omega_2^3} \end{pmatrix} = R_\theta \begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix}$$

$$\omega^1\otimes\omega_1^3 + \omega^2\otimes\omega_2^3 = \overline{\omega^1}\otimes\overline{\omega_1^3} + \overline{\omega^2}\otimes\overline{\omega_2^3}$$

$\square$

注记. Weingarten变换 W 在 $\{e_1, e_2\}$ 下的矩阵表示

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix}$$

如果 $\{e_1, e_2\}$ 在每一点处均为主方向，则

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

$$\begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = \begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} k_1\omega^1 \\ k_2\omega^2 \end{pmatrix}$$

$\Rightarrow \mathrm{II} = \underline{k_1}\,\omega^1\otimes\omega^1 + \underline{k_2}\,\omega^2\otimes\omega^2$ 主曲率

脐点

Principle frame field.

Principle frame s

Claim: M 正则曲面片, $p \in M$ 不是脐点, 则存在 p 点之邻域 U s.t. U 上存在正交活动标架 $\{r; e_1, e_2, e_3\}$ 满足 e_1, e_2 为主方向 □

ω_1^2 是什么?