

作业 16 (指标计算)

定义 $R_{\alpha\beta\gamma} := -g_{\delta\epsilon} \left(\frac{\partial \Gamma_{\alpha\beta}^{\delta}}{\partial u^{\gamma}} - \frac{\partial \Gamma_{\alpha\gamma}^{\delta}}{\partial u^{\beta}} + \Gamma_{\alpha\beta}^{\eta} \Gamma_{\eta\gamma}^{\delta} - \Gamma_{\alpha\gamma}^{\eta} \Gamma_{\eta\beta}^{\delta} \right)$

<1> 证明:

$$R_{\alpha\beta\gamma} = \frac{1}{2} \left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^{\delta} \partial u^{\alpha}} + \frac{\partial^2 g_{\alpha\beta}}{\partial u^{\delta} \partial u^{\delta}} + \frac{\partial^2 g_{\delta\gamma}}{\partial u^{\beta} \partial u^{\alpha}} - \frac{\partial^2 g_{\alpha\gamma}}{\partial u^{\beta} \partial u^{\delta}} \right) + \Gamma_{\alpha\beta}^{\eta} \Gamma_{\eta\gamma}^{\delta} - \Gamma_{\alpha\gamma}^{\eta} \Gamma_{\eta\beta}^{\delta},$$

其中 $\Gamma_{\eta\delta\gamma} = g_{\eta\epsilon} \Gamma_{\delta\gamma}^{\epsilon}$ 为第二类 Christoffel 符号

<2> 计算所有 Christoffel 符号 $\{\Gamma_{\beta\gamma}^{\alpha}, \alpha, \beta, \gamma = 1, 2\}$ 用 E, F, G 的表达式。(比如 $\Gamma_{11}^1 = \frac{1}{EG-F^2} \left(\frac{G}{2} E_u + \frac{E}{2} E_v - FF_u \right)$)

<3> 应用 <1> 和 <2> 证明:

$$4(EG-F^2) R_{1212} = E(E_v G_v - 2F_u G_v + (G_v)^2) + F(E_u G_v - E_v G_u - 2E_v F_v + 4F_u F_v - 2F_u G_u) + G(E_u G_u - 2E_u F_v + (E_v)^2) - 2(EG-F^2)(E_{vv} - 2F_{uv} + G_{uu})$$

(与高斯绝妙定理比较回忆 Gauss 方程告诉我们 $R_{1212} = K(EG-F^2)$)

<4> 设 $r: D \rightarrow \mathbb{R}^3, (u, v) \mapsto r(u, v)$ 为一正则曲面片且其参数化满足 $F=M=0$. 应用 <2> 证明 Codazzi 方程组

$$\begin{cases} \frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} = \Gamma_{12}^3 b_{31} - \Gamma_{11}^3 b_{32} \\ \frac{\partial b_{21}}{\partial u^2} - \frac{\partial b_{22}}{\partial u^1} = \Gamma_{22}^3 b_{31} - \Gamma_{21}^3 b_{32} \end{cases}$$

用记号 E, F, G, L, M, N 可表为 $L_v = H E_v, N_u = H G_u, H$ 为平均曲率。