HOMEWORK 1: GEODESICS

RIEMANNIAN GEOMETRY, SPRING 2022

1. (Christoffel symbols)

Let $(U, x = (x^1, \dots, x^n))$ be a chart of a Riemannian manifold M. Let

$$(x^1, \dots, x^n) \to (y^1, \dots, y^n)$$

be a smooth coordinate change, and the Riemannian metric can be written as $g_{ij}(x)dx^i \otimes dx^j$ and $h_{\alpha\beta}(y)dy^\alpha \otimes dy^\beta$ respectively.

(i) Show the transformation formula of g^{ij} under the coordinate change is

$$g^{ij}(x) = h^{\alpha\beta}(y(x))\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}.$$

(ii) Compute the transformation formulae of the Christoffel symbols Γ^i_{jk} under the coordinate change. Do they define a tensor?

(iii) Let $\gamma : [a, b] \to U$ be a smooth curve. Denote $\dot{x}^i(t) := \frac{d}{dt}x^i(\gamma(t))$. Compute the transformation formula of

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)$$

under the coordinate change.

Remark: *Elwin Christoffel* (1829-1900) was noted for his work in mathematical analysis, in which he was a follower of *Dirichlet* and *Riemann*. He wrote important papers which contributed to the development of the tensor calculus of *Gregorio Ricci-Curbastro* and *Tullio Levi-Civita*. The *Christoffel symbols* which he introduced are fundamental in the study of tensor analysis. The *Christoffel reduction theorem*, so named by *Klein*, solves the *local equivalence problem for two quadratic differential forms*. *Paul Butzer* once commented:

> *The procedure Christoffel employed in his solution of the equivalence problem is what Gregorio Ricci-Curbastro later called* **covariant differentiation***, Christoffel also used the latter concept to define the basic* **Riemann-Christoffel curvature tensor**. ... *The importance of this approach and the two concepts Christoffel introduced, at least implicitly, can only be judged when one considers the influence it has had.*

Indeed this influece is clearly seen since this allowed *Ricci-Curbastro* and *Levi-Civita* to develop a coordinate free differential calculus which *Einstein*, with the help of *Grossmann*, turned into the tensor analysis mathematical foundation of general relativity.
(*Read more at* http://mathshistory.st-andrews.ac.uk/Biographies/Christoffel.html)

2. (Critical point of energy functional)

Let $\Omega \subset \mathbb{R}^m$ be an open domain with the Riemannian metric given by the matrix $(g_{ij}(x))$, $x \in \Omega$. Let $U \subset \mathbb{R}^n$ be an open domain with the Riemannian metric given by the matrix $(h_{\alpha\beta}(y))$, $y \in U$. Let $f : \Omega \to U$ be a smooth map which is a critical point of the following energy functional:

$$E(f) := \frac{1}{2}\int_\Omega g^{ij}(x)h_{\alpha\beta}(f(x))\frac{\partial f^\alpha(x)}{\partial x^i}\frac{\partial f^\beta(x)}{\partial x^j}\sqrt{g(x)}dx,$$

where we use the notation that

$$f(x) = f(x^1, \dots, x^m) = (f^1(x^1, \dots, x^m), \dots, f^n(x^1, \dots, x^m)),$$

and $g(x) = \det(g_{ij}(x))$, dx being the Lebesgue measure. Such a map f is called a *harmonic map* from (Ω, g) to (U, h).

(i) Compute the differential equations that a harmonic map $f : \Omega \to U$ has to satisfy.

(ii) Show that a geodesic in (U, h) is a harmonic map.

(ii) If $U = \mathbb{R}$ with the Euclidean metric, then a harmonic map $f : \Omega \to U$ is called a *harmonic function* on (Ω, g). Write down the differential equations that harmonic functions satisfy.

Remark: Harmonic maps between Riemannian manifolds are canonical objects from the points of view of topology and calculus of variations. These maps provide a rich display of both differential geometric and analytic phenomena. Much of the study of these maps serves as a model for many other challenging problems in geometric analysis and has been athe source of inspiration and undiminishing fascination. ... Harmonic maps with two dimensional domains present special features that are crucial for applications to minimal surfaces (i.e., conformal harmonic maps) and to the deformation theory of Riemann surfaces-Teichmüller theory.... [Preface of "Harmonic maps and their heat flows" by Fanghua Lin and Changyou Wang.]

Harmonic maps into spheres or complex projective spaces have also acquired some physical interest since they turned out to be solutions of the nonlinear $O(N)$ σ-models. For more details, we refer to Misner, Harmonic Maps as Models for Physical Theories, Phys. Rev. D 18 (12) (1978). [Section 1.5 of "Harmonic maps between surfaces" by Jürgen Jost.]

3. (Hyperbolic spaces)

Recall that the **hyperboloid**

$$H^n := \left\{(x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n (x^i)^2 - (x^{n+1})^2 = -1, x^{n+1} > 0\right\}$$

is a differentiable manifold with the following chart:

$$y : \ H^n \longrightarrow B_1(0) := \left\{(y^1, \dots, y^n) \in \mathbb{R}^n : \sum_{i=1}^n (y^i)^2 < 1\right\} \subset \mathbb{R}^n,$$

$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y^1, \dots, y^n) := \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}}\right).$$

Let g be the Riemannian metric of H^n given by

$$g_{ij} = \frac{4}{(1 - \sum_{i=1}^n (y^i)^2)^2}\delta_{ij}.$$

(i) Compute the Christoffel symbols and write down the system of differential equations satisfied by the geodesics.

(ii) Determine the geodesics of H^n through the point $(0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ (whose coordinate is $(0, \dots, 0) \in B_1(0) \subset \mathbb{R}^n$).

(iii) Is H^n a complete Riemannian manifold?

Hint: We point out the following useful fact: The function

$$y(t) := \frac{e^t - 1}{e^t + 1}, \ t \in [0, \infty)$$

is a solution of the following ODE:

$$\begin{cases} \ddot{y}(t) + \frac{2y(t)}{1-y(t)^2}\dot{y}(t)^2 = 0, \\ y(0) = 0. \end{cases}$$

4. (Completeness)

(i) Assume that (M, g) has the property that all normal geodesics exist for a fixed time $\epsilon > 0$. Show that (M, g) is geodesically complete.

(ii) Let (M, g) be a metrically complete Riemannian manifold and $\tilde{g}$ is another metric on M such that $\tilde{g} \geq g$. Show that $(M, \tilde{g})$ is also metrically complete.

(iii) Let (M, g) be a Riemannian manifold which admits a proper Lipschitz function $f : M \to \mathbb{R}$. Show that (M, g) is complete. (Recall that a function between topological spaces is called proper if inverse images of compact subsets are compact.)

Remark: The Hopf-Rinow Theorem is named after German mathematicians *Heinz Hopf* (1894-1971) and his student *Willi Rinow* (1907-1979), who published it in *Über den Begriff der vollständigen differentialgeometrischen Fläche, Commentarii Mathematici Helvetici, **3**, 209-225 (1931)* (The title in English: *On the concept of complete differentiable surfaces*). The Hopf-Rinow theorem is generalized to length-metric spaces in the following way: If a length-metric space (M, d) is *complete* and *locally compact* then any two points in M can be connected by a shortest geodesic, and any bounded closed set in M is compact. The theorem does **not** hold in infinite dimensions: *C. J. Atkin (The Hopf-Rinow Theorem is false in infinite dimensions, Bulletin of the London Mathematical Society, **7**(3), 261-266, 1975)* showed that two points in an infinite dimensional complete Hilbert manifold need not be connected by a geodesic (even if you do not require this geodesic to be a shortest curve).

Heinz Hopf worked on the fields of topology and geometry. In his dissertation, *Über Zusammenhänge zwischen Topologie und Metrik von Mannigfaltigkeiten* in 1925 (in English, *connections between topology and metric of manifolds*), he proved that any simply connected *complete* Riemannian 3-manifold of constant sectional curvature is globally isometric to *Euclidean, spherical, or hyperbolic space.* He also studied the indices of zeros of vector fields on hypersurfaces, and connected their sum to curvature. Some six months later he gave a new proof that the sum of the indices of the zeros of a vector field on a manifold is independent of the choice of vector field and equal to the *Euler characteristic* of the manifold. This theorem is now called the *Poincaré-Hopf theorem.* Hopf spent the academic year 1927/28 at Princeton University and at this time he discovered the *Hopf invariant* of maps $S^3 \to S^2$ and proved that the *Hopf fibration* has invariant 1.