HOMEWORK 2: CONNECTIONS AND CURVATURE

RIEMANNIAN GEOMETRY, SPRING 2022

1. Torsion tensor

Let M be a smooth manifold. Let ∇ and $\overline{\nabla}$ be two affine connections on M. We define

$$D(X,Y) := \nabla_X Y - \overline{\nabla}_X Y, \ \forall\ X, Y \in \Gamma(TM).$$

(i) Prove that D is a tensor, that is, D is linear over C^∞ functions in both arguments.

(ii) Prove that there is a unique way to write

$$D = S + A$$

with S *symmetric* and A *alternating*, i.e., $S(X,Y) = S(Y,X)$ and $A(X,Y) = -A(Y,X)$.

(iii) Prove that ∇ and $\overline{\nabla}$ have the same torsion if and only if $A = 0$.

(iv) A parametrized curve $\gamma = \gamma(t)$ on M is called *a geodesic with respect to an affine connection* ∇ if $\left(\nabla_{\dot\gamma(t)}\dot{\gamma(t)}\right)(\gamma(t)) = 0$ for any t.

Prove that the following are equivalent:

(a) ∇ and $\overline{\nabla}$ have the same geodesics;
(b) $D(X,X) = 0$ for any $X \in \Gamma(TM)$.
(c) $S = 0$.

(v) Prove that if ∇ and $\overline{\nabla}$ have the same geodesics and the same torsion, then $\nabla = \overline{\nabla}$.

(vi) Prove that for any affine connection ∇ on M, there exists a unique affine connection $\overline{\nabla}$ with the same geodesics and with torsion 0. (*Hint*: Consider the connection $\overline{\nabla}_X Y := \nabla_X Y - \frac{1}{2}T(X,Y)$, where T is the torsion of ∇.)

Remark: We can define two affine connections with the same geodesics to be equivalent. Then all affine connections on M can be divided into equivalent classes. The above discussions tell us that each equivalent class has *exactly one* connection with zero torsion.

2. Connections on spheres

Let S^n be the sphere with the induced metric g from the Euclidean metric in $\mathbb{R}^{n+1}$. We denote by $\overline{\nabla}$ the canonical Levi-Civita connection on $\mathbb{R}^{n+1}$. For any $X, Y \in \Gamma(T\mathbb{S}^n)$, one can extend X, Y to smooth vector field $\overline{X}, \overline{Y}$ on $\mathbb{R}^{n+1}$, at least near $\mathbb{S}^n$.

By locality, the vector $\overline{\nabla}_{\overline{X}}\overline{Y}$ at any $p \in \mathbb{S}^n$ depends only on $\overline{X}(p) = X(p)$ and the vectors $\overline{Y}(q) = Y(q)$ for $q \in \mathbb{S}^n$. That is, $\overline{\nabla}_{\overline{X}}\overline{Y}$ is independent of the extension of X, Y we choose. So we will write $\overline{\nabla}_X Y$ instead of $\overline{\nabla}_{\overline{X}}\overline{Y}$ at points on $\mathbb{S}^n$.

We define $\nabla_X Y$ to be the orthogonal projection of $\overline{\nabla}_X Y$ onto the tangent space of $\mathbb{S}^n$, i.e.,

$$\nabla_X Y := \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \mathbf{n}\rangle \mathbf{n},$$

where $\mathbf{n}$ is the unit out normal vector on $\mathbb{S}^n$.

(i) Prove that ∇ is an affine connection on $\mathbb{S}^n$.

(ii) Prove that ∇ is the Levi-Civita connection of $(\mathbb{S}^n, g)$.

Remark: This is in fact a general way to construct the Levi-Civita connection on a Riemannian manifold. Recall that any Riemannian manifold (M,g) can be embedded isometrically to a Euclidean space E of large enough dimension. For any $p \in M$, we have the orthogonal projection map

$$\pi(p) : T_pE \to T_pM.$$

The composition of this orthogonal projection map with the canonical Levi-Civita connection on E (given by directional derivatives) produces the Levi-Civita connection on (M, g).

3. A natural extension of Gauss' lemma

Let N_1, N_2 be two submanifolds of a complete Riemannian manifold (M, g), and let $\gamma : [0, a] \to M$ be a geodesic such that $\gamma(0) \in N_1$, $\gamma(a) \in N_2$ and γ is the shortest curve from N_1 to N_2. Prove that $\dot\gamma(0)$ is perpendicular to $T_{\gamma(0)}N_1$, and $\dot\gamma(a)$ is perpendicular to $T_{\gamma(t)}N_2$.

4. (Hyperbolic spaces)

Recall that the **hyperboloid**

$$H^n := \left\{ (x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n (x^i)^2 - (x^{n+1})^2 = -1, x^{n+1} > 0 \right\}$$

is a differentiable manifold with the following chart:

$$y : \; H^n \longrightarrow B_1(0) := \left\{ (y^1, \ldots, y^n) \in \mathbb{R}^n : \sum_{i=1}^n (y^i)^2 < 1 \right\} \subset \mathbb{R}^n,$$

$$(x^1, \ldots, x^n, x^{n+1}) \mapsto (y^1, \ldots, y^n) := \left(\frac{x^1}{1 + x^{n+1}}, \ldots, \frac{x^n}{1 + x^{n+1}} \right).$$

Let g be the Riemannian metric of H^n given by

$$g_{ij} = \frac{4}{(1 - \sum_{i=1}^n (y^i)^2)^2} \delta_{ij}.$$

(i) Compute the sectional curvature, Ricci curvature, and scalar curvature of the hyperbolic space H^n. (Recall we have computed the Christoffel symbols in Excercise 1.)

5. (The Second Variation Formula for length)

Let $\gamma : [a, b] \to M$ be a smooth curve and

$$F : [a, b] \times (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$$

be a 2-parameter variation of γ. Denote by

$$V(t) := \frac{\partial F}{\partial v}(t, 0, 0), \quad W(t) = \frac{\partial F}{\partial w}(t, 0, 0)$$

the two corresponding variational fields. Let $L(v, w) := L(\gamma_{v,w})$ be the length of the curve $\gamma_{v,w}(t) := F(t, v, w), t \in [a, b]$.

(1) Show that

$$\frac{\partial^2}{\partial w \partial v} L(v,w) = \int_a^b \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|} \Big\{ \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w} \right\rangle - \left\langle R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right) \frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle + \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle - \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|^2} \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle \Big\} dt,$$

where $\left\| \frac{\partial F}{\partial t} \right\| := \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle^{\frac{1}{2}}$.

(2) Let γ be a normal geodesic. Show that

$$\frac{\partial^2}{\partial w \partial v}_{|v=w=0} L(v,w) = \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle - T\langle V,T \rangle T \langle W,T \rangle \right) dt + \langle \nabla_W V, T \rangle |_a^b,$$

where $T(t) := \dot{\gamma}(t)$ is the velocity field along γ.

(3) Consider the orthogonal component $V^\perp, W^\perp$ of V, W with respect to T, that is

$$V^\perp := V - \langle V,T \rangle T,$$
$$W^\perp := W - \langle W,T \rangle T.$$

Show that

$$\frac{\partial^2}{\partial w \partial v}_{|v=w=0} L(v,w) = \int_a^b \left(\langle \nabla_T V^\perp, \nabla_T W^\perp \rangle - \langle R(W^\perp,T)T, V^\perp \rangle \right) dt + \langle \nabla_W V, T \rangle |_a^b,$$

6. (Existence of Riemmannian metrics with positive curvature)
Consider the smooth manifold $RP^n \times RP^n$.

(i) Does there exist a Riemannian metric on $RP^n \times RP^n$ with positive sectional curvature?
(ii) Does there exist a Riemannian metric on $RP^n \times RP^n$ with positive Ricci curvature?

Remark: It is completely unknown whether $S^2 \times S^2$ has a Riemannian metric with positive sectional curvature or not. This is known as the Hopf problem.