HOMEWORK 3: JACOBI FIELDS

RIEMANNIAN GEOMETRY, SPRING 2022

1. (Taylor expansions)

Let (M^n, g) be a Riemannian manifold. Let $\gamma : [0, b] \to M$ be a normal geodesic with $\gamma(0) = p$ and $\gamma'(t) =: T(t)$.

(1) Let J_1, J_2 be Jacobi fields along γ with $J_i(0) = 0,\ i = 1, 2$. Denote

$$w_i := \nabla_T J_i(0),\ i = 1, 2.$$

Show the following Taylor expansion for small t:

$$\langle J_1(t), J_2(t)\rangle = \langle w_1, w_2\rangle t^2 - \frac{1}{3}\langle R(w_1, \gamma'(0))\gamma'(0), w_2\rangle t^4 + O(t^5).$$

(2) Let $e_1, \dots, e_n$ be an orthonormal basis of T_pM. Let us denote by

$$g_{jk}(t) := \langle (d\exp_p)_{t\gamma'(0)}(e_j), (d\exp_p)_{t\gamma'(0)}(e_k)\rangle,\ t \in (0, b].$$

Show the following Taylor expansion for small t:

$$g_{jk}(t) = \delta_{jk} - \frac{t^2}{3}\langle R(e_j, \gamma'(0))\gamma'(0), e_k\rangle + O(t^3).$$

(3) Consider the ball $B_p(r) := \{x \in M : d(p, x) < r\}$. Show the following Taylor expansion for small r:

$$\mathrm{vol}(B_p(r)) = \omega_n r^n - \frac{r^{n+2}}{6(n+2)}\int_S \mathrm{Ric}(u, u)du + O(r^{n+3}),$$

where ω_n denotes the volume of unit ball in the n dimensional Euclidean space, the integration is over the unit sphere, S in T_pM (in its Euclidean metric) and du is the standard volume measure on S.

2. (Length of the geodesic circle)

Consider a simply-connected complete two dimensional Riemannian manifold (M^2, g) with Gauss curvature $\leq \beta$.

Given $O \in M$. Denote by $c(r)$ the length of the curve

$$\{x \in M : d(x, O) = r\}.$$

Show that for any $r \geq 0$,

$$c(r) \geq \begin{cases} 2\pi r, & \text{if } \beta = 0; \\ \frac{2\pi}{\sqrt{-\beta}}\sinh\sqrt{-\beta}r, & \text{if } \beta < 0. \end{cases}$$

3. (Laplacian comparison theorem)

Let (M^n, g) be an n-dimensional complete Riemannian manifold with $Ric \geq 0$, and $p_0 \in M$ be a given point.

(1) Let $C(p_0) \subset M$ be the cut locus of p_0. For any $p \in C(p_0)$, let γ be a normal geodesic with $\gamma(0) = p_0$ and $\gamma(d(p_0, p)) = p$. Let $p_\epsilon := \gamma(\epsilon)$ for a small $\epsilon > 0$. Show that $p \notin C(p_\epsilon)$.

(2) Recall that an **upper barrier** for a continuous function f at a point $x_0 \in M$ is a C^2 function $g : M \to \mathbb{R}$, defined in a neighborhood of x_0 such that

$$g(x_0) = f(x_0) \text{ and } g(x) \geq f(x), \text{ in the neighborhood.}$$

For a continuous function f, we say that $\boldsymbol{\Delta}\mathbf{f} \leq \mathbf{a}$ **at** x_0 **in the barrier sense**, if for any $\epsilon > 0$, there is an upper barrier $f_{x_0,\epsilon}$ of f at x_0, such that

$$\Delta f_{x_0,\epsilon} \leq a + \epsilon.$$

Consider the function $\rho : M \to \mathbb{R}$, $\rho(x) := d(x, p_0)$. Show that

$$\Delta\rho \leq \frac{n-1}{\rho}$$

at any $p \in C(p_0)$ in the barrier sense.

4. (Günther 1960)

Let (M^n, g) be a complete Riemannian manifold with sectional curvature $\leq k$ for some $k \in \mathbb{R}$. Let $B_p(r) := \{x \in M : d(p, x) < r\}$ be a ball in M which does not meet the cut locus of p. Show that

$$\text{vol}(B_p(r)) \geq \text{vol}(B^k(r)),$$

where $B^k(r)$ is a ball with radius r in a simply-connected space form with constant sectional curvature k.