What is "nice" metric?

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Geometric analysts always want to find a "nice" metric on a given manifold, which seems to be canonical. The definition of "nice" might be a subtle problem. A natural idea is that it makes some common functional, such as scalar curvature functional, achieve its minimum.

Yamabi's Canonical Metric in Riemannian Geometry: Scalar curvature is a higher dimensional analogue of Gauss curvature. Yamabe set up a problem that if we can find a metric such that its scalar curvature equal to a given smooth function, moreover, in a given conformal class. In [Yam60], he attempted to show that any Riemannian structure on a compact manifold of dimension not less than 3 could be pointwise conformally deformed to one of constant scalar curvature. Trudinger [Tru68] pointed out a serious gap in Yamabe's proof, and the assertion is in doubt. Kazdan and Warner [KW73] had proved that, as long as the function is negative somewhere, there is a metric whose scalar curvature coincide with the given function. With this in mind, finding a metric whose scalar curvature is positive at every point sounds easy, since it is only one scalar inequality on the entire metric. However, there is a topological obstruction to the existence of metrics with positive scalar curvature.

Actually, Some manifold can not have metric with positive scalar curvature everywhere. For example, the torus T^n does not admit any metric with positive scalar curvature everywhere [SY79] [GL80b]. In dimension 2 [KW74], we have considered the problem of Gaussian curvature on 2-manifolds. The key to our study of Gaussian curvatures was the Gauss-Bonnet theorem which imposes sign restrictions on the Gaussian curvatures of compact 2-manifolds depending on the Euler characteristic. There is also a topological implication of scalar curvature which provides an obstruction to positive scalar curvature for certain special manifolds. Lichnerowicz has shown [Lic63] that if the scalar curvature is nonnegative, but not identically zero, on a compact even-dimensional spin manifold, then there are no harmonic spinors. From this fact, using the Atiyah-Singer index theorem [AS68] he concluded that the Hirzebruch $\hat{A}$ genus of such a manifold must be zero. Thus one cannot have a metric with nonnegative scalar curvature, except possibly identically zero, on a compact spin manifold whose $\hat{A}$ genus is not zero. Examples of such manifolds arise in the theory of spin cobordism, see [Mil65].

There is a generalization of $\hat{A}$ which has been shown to completely characterize when a (simply connected, spin) manifold admits a metric of positive scalar curvature. It is usually denoted $\alpha(M)$, and was first introduced in 1974 by Hitchen [Hit74] who showed that if there is a metric with positive scalar curvature, then $\alpha(M) = 0$. The converse was established by Stolz in 1990 [Sto90]. For our purposes, its most important property is that it can be nonzero only in dimensions $n = 0, 1, 2, 4 \bmod 8$. (This assumes $n > 4$.) So there is a possible obstruction to positive scalar curvature only in these dimensions. Even in these dimensions, this obstruction only applies to spin manifolds: If M does not admit spinors, then it always admits a metric of positive scalar curvature [GL80a].

In view of Yamabe's problem, the "nice" metric can be thought of witch has constant scalar curvature. It seems to be special. But the constant can not be arbitrary. Yabame invariant $\mu(g)$, which is a conformal invariance characterize this constant. Nontrivial solution of its Euler-Lagrange equation

$$-4\frac{n-1}{n-2}\Delta_g f + S(g)f = \mu(g)f^{\frac{n+2}{n-2}} \tag{1}$$

whose existence guaranteed by the solution of Yamabe problem [Sch84], gives rise to the so-called Yamabe metric $f^{\frac{4}{n-2}}g$, which has constant scalar curvature $\mu(g)$.

Calabi's Canonical Metric in Kähler Geometry: In 1950s, Calabi first proposed to study the constant scalar curvature Kähler (cscK) metric problems. His ideal is to find the best canonical metric in each given Kähler class [Cal82] [Cal85], which results in a 4th order, fully nonlinear partial differential equation. The relative PDE is very difficult for one cannot use maximal principle from PDE point of view and also can not have much control of metric from the bound of the scalar curvature. When the first Chern class has a definite sign (positive, negative or zero), the cscK metric in the suitable multiple of the first Chern class reduces to a Kähler-Einstein metrics, which is the center of the field for the last few decades where all efforts and techniques of many mathematicians are devoted to, leading to the final resolution of this difficult problem.

In 1958, E. Calabi published the fundamental C^3 estimate for Monge-Ampére equation [Cal58] which later played a crucial role in Yaus seminal resolution of Calabi conjecture [Yau78] in 1976 when the first Chern class is either negative or zero (In negative case, T. Aubin has an independent proof [Aub76]) . This work of Yau is so influential that generations of experts in Kähler geometry afterwards largely followed the same route: Securing a C^0 estimate first, then move on to obtain C^2, C^3 estimates etc. In the case of positive first Chern class, Gang Tian proved Calabi conjecture in 1989 [Tia89] for Fano surfaces when the automorphism group is reductive. It is well known that there are obstructions to the existence of KE metrics in Fano manifolds; around 1980s, Yau proposed a conjecture which relates the existence of Kähler Einstein metrics to the stability of underlying tangent bundles. In 1997, Gang Tian introduced the so-called K-stability (via special degeneration) and showed that the existence of Kähler-Einstein metric necessarily implies the K-stability of the underlying polarization through special degeneration [Tia97]. In 2002, S. K. Donaldson reformulated it into a notion of algebraic K-stability [Don02]. This conjecture was settled in 2012 through a series of work [CDS12a] [CDS12b] [CDS12c], which is itself quite involved as it sits at the intersection of several different subjects: algebraic geometry, several complex variables, geometry analysis and metric differential geometry etc. With the existence problem of Kähler-Einstein metric settled eventually, the next step is to discuss how to attack Calabis original problem in full generality.

Conjecture 0.1 (Yau-Tian-Donaldson) *Polarized Kähler manifold (M,L) is K-stable if and only if there exist a cscK metric in* $c_1(L)$*.*

In 2015, Chen Xiuxiong propose a "new" continuity path in a given Kähler class to solve the cscK metric problem [Che15]. Also in 2018, Chen Xiuxiong and Cheng Jingrui derived apriori estimates for constant scalar curvature Kähler metrics on a compact Kähler manifold, and proved Donaldsons conjecture on the equivalence between geodesic stability and existence of cscK when $Aut_0(M, J) \neq 0$ [CC18a] [CC18b] [CC18c]. This deep result generalizes Tian Gang's Properness theorem, the Mabuchi energy is proper if and only if there is a metric of constant scalar curvature in the class $[\omega]$. On the other hand, Sean Paul gives a complete description of the behavior of the Mabuchi energy along all degenerations. Under the assumption that $Aut(M, J)$ is finite, this gives the equivalent between analytic stability and algebraic stability.

Theorem 0.1 *Let* (X, L) *be an arbitrary polarized manifold. Assume that* $Aut(M, J)$ *is finite. Then* (X, L) *is asymptotically K-stable if and only if there is a constant scalar curvature metric in* $c_1(L)$*.*

The most important idea is to identify a norms conformally equivalent to the standard L^2 norms on polynomials. Since the conformal factors are continuous, they are bounded by reasons of compactness. The conclusion was that the Mabuchi energy is almost the distance between the orbits in Hilb. That is, the distance in the usual Fubini Study metric induced by L^2 up to some (unknown) error that depended on the degree of the embedding. Based on work by J.M.Bismut, Henri Gillet, and Christophe Soulé [BGS88a] [BGS88b] [BGS88c] , Paul [Pau12a] recently found a more sophisticated path to the relationship between the Mabuchi energy restricted to the Bergman metrics and the resultant and hyperdiscriminant of the subvariety which revealed that the error was in fact the difference between the L^2 norm and another well known L^0 norm, i.e. the Mahler measure. The boundedness of the error, initially attributed to compactness, is just an expression of the fact that these norms are comparable. The outcome is that the norm on the space of polynomials which connects the Mabuchi energy to stability of the pair (R, Δ) is exactly given by the Mahler measure. Now asymptotic stability and global bounds on K-energy maps follow almost at once from Tian's Thesis [Tia90].

There is also another approach, called Kähler-Ricci flow, to study the existence problem of Kähler-Einstein metrics on Fano manifolds. In general two key ingredients are needed, namely the partial C^0-estimate and the construction of a destabilizing test configuration. The first is analytic and the second is algebraic in nature. For the partial C^0-estimate, it is proved by Székelyhidi [Szé13] for the classical Aubin-Yau continuity path, by adapting the results of [DS14] [CDS12b] [CDS12c]. For the approach using Ricci flow, this is proved by Chen-Wang [CW12] in dimension two, Tian-Zhang [TZ13] in dimension three, and by Chen-Wang [CW14] in all dimensions as a consequence of the resolution of the Hamilton-Tian conjecture. We note that these results together with the work of Sean Paul [Pau12a] [Pau12b] [Pau13] already imply that on a Fano manifold without non-trivial holomorphic vector fields, the existence of a Kähler-Einstein metric is equivalent to the notion of stability defined by Paul. About the second ingredient, Datar and Szëkelyhidi [DS15a] have adapted the results of [CDS12c] to the Aubin-Yau continuity path, which gives a new proof of the theorem of Chen-Donaldson-Sun. Chen-Wang [CSW15] give yet another proof using the Ricci flow, which means that technically they will address the issue of constructing a de-stabilizing test configuration. Notice this can not be naively adapted from [CDS12c] and requires new strategy to understand the relation between the asymptotic behavior of the Kähler-Ricci flow and algebraic geometry. Their work is motivated by [DS15b] which studies tangent cones of non-collapsed Kähler-Einstein limit spaces.

When consider about cscK metrics, there is a flow, called Calabi flow which is supposed to be used to get cscK metric. Motived by Donaldsons theorem relating balanced embeddings to metrics with constant scalar curvature [Don01]. Joel Fine [Fin10] prove the parabolic analogue, balancing flow, which can approximate the Calabi flow using Donaldsons techniques [Don09] with an asymptotic result of Liu and Ma [LM07]. But the long time existence problem of Calabi flow is still open.

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