Riemannian geometry Lecture 15

Sectional curvature (M,g), $p\in M$, $\Pi_p \subset T_pM$ 2-dim subspace $= \operatorname{span}\{X_p, Y_p\}$

$$K(\Pi_p) := \frac{R(X_p, Y_p, X_p, Y_p)}{G(X_p, Y_p, X_p, Y_p)} \qquad G(X_p, Y_p, X_p, Y_p) \longrightarrow \left(\text{area}\left(\quad\right)\right)^2$$

Ricci curvature tensor (0,2)-tensor symmetric

$$Ric(Y,Z) := \operatorname{tr} R(\cdot, Y, \cdot, Z)$$

$$= \operatorname{tr}\left(X \mapsto R(X,Z)Y\right)$$

$\hookrightarrow$ a linear transformation $T_pM \longrightarrow T_pM$.

$$\Rightarrow Ric(Y,Z) = Ric(Z,Y)$$

p (U,x) chart $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$

$$Ric_{pq} := Ric\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^q}\right) = \operatorname{tr}\left(\underline{X \mapsto R\left(X, \frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial x^q}}\right)$$

$$\left(\frac{\partial}{\partial x^i} \mapsto R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial x^q} = \boxed{R^k{}_{qip}}\frac{\partial}{\partial x^k}\right)$$

$$Ric_{pq} = \sum_{i=1}^n R^i{}_{qip} = \sum_{l,i=1}^n \delta^i_l R^l{}_{qip} = \sum_{l,i} g^{ji} \underline{g_{jl} R^l{}_{qip}}$$

$$= \sum_{i,j} g^{ij} R_{jqip}$$

$$\begin{cases} Ric(X,Y) = Ric(Y,X) & (0,2)\text{-tensor} \\ g(X,Y) = g(Y,X) & (0,2)\text{-tensor} \end{cases} \Rightarrow \underline{Ric(X,X) = 0,\ \forall X} \Rightarrow \boxed{Ric(X,Y) = 0,\ \forall X, Y}$$

$$0 = Ric(X+Y, X+Y) = \underset{\shortparallel}{Ric(X,X)} + \underbrace{Ric(X,Y)} + \underbrace{Ric(Y,X)} + \underset{\shortparallel}{Ric(Y,Y)}$$

$$0 = \mathrm{Ric}(X+Y, X+Y) = \underset{\overset{\shortparallel}{0}}{\mathrm{Ric}(X,X)} + \underbrace{\mathrm{Ric}(X,Y) + \mathrm{Ric}(Y,X)}_{=2\,\mathrm{Ric}(X,Y)} + \underset{\overset{\shortparallel}{0}}{\mathrm{Ric}(Y,Y)}$$

$$\Rightarrow \mathrm{Ric}(X,Y) = 0, \ \forall X, Y$$

$$\mathrm{Ric}(X,X)(p) = \mathrm{Ric}(X_p, X_p)$$

$$\frac{\mathrm{Ric}(X,X)}{g(X,X)} = \mathrm{Ric}\left(\frac{X}{\sqrt{g(X,X)}}, \frac{X}{\sqrt{g(X,X)}}\right)$$

Definition The Rie mfld (M,g) is called an Einstein manifold with Einstein constant k, if

$$\mathrm{Ric}(X,X) = k\, g(X,X), \quad \forall X \in \Gamma(TM).$$

i.e. M has "constant Ricci curvature".

Prop. A Rie. manifold is an Einstein manifold with Einstein constant k iff $\mathrm{Ric} = k g$

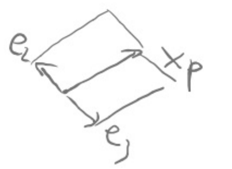
Pf. $S := \mathrm{Ric} - kg$ symmetric $(0,2)$-tensor

$$S(X,X) = 0, \ \forall X \Rightarrow S(X,Y) = 0, \ \forall X, Y. \quad \square$$

Rmk: $p \in M$, $X_p \in T_pM$. unit tangent vector.

Extend it to be an orthonormal basis $\{\overset{e_1}{\overset{\shortparallel}{X_p}}, e_2, \ldots, e_n\}$ of T_pM. Then

$$\begin{aligned} \mathrm{Ric}(X_p, X_p) &= \mathrm{tr}\, R(\cdot, X_p, \cdot, X_p) \\ &= \sum_{i=2}^{n} R(e_i, X_p, e_i, X_p) \\ &= \sum_{i=2}^{n} K(\Pi_{\{e_i, X_p\}}) \end{aligned}$$



If (M,g) has constant sectional curvature k, then (M,g) is Einstein with constant $\underline{(n-1)k}$.

Thm (Schur) Let (M,g) be a connected Rie mfld of $n \geq 3$

If $\mathrm{Ric}(X_p, X_p) = f(p)\, g(X_p, X_p) \quad \forall X_p \in T_pM.$

where $f(p)$ depends only on p

Then $f(p) \equiv$ constant. $\forall p \in M$.

Then $f(p) \equiv \underline{\text{constant}}$. $\forall p \in M$.

Rmk ① If $K(\Pi_p) = f(p)$ depends only on p,

then $\dfrac{Ric(X_p, X_p)}{g(X_p, X_p)} = (n-1) f(p)$ depends only on p

$\overset{\text{Thm}}{\Longrightarrow}$ $(n-1) f(p) \equiv$ constant $\Rightarrow f(p) \equiv$ const. $\square$

② By assumption, we have $Ric = f\, g$

Taking covariant derivative $\underline{\nabla_v Ric} = \nabla_v (f\, \underline{\underline{g}})$
$$= v(f)\, g$$

Proof. $\forall p$. $\boxed{X_p(f) = 0, \ \forall X_p \in T_pM}$

Pick a normal coordinate (U, x) around p

$$\underline{Ric = f\, g} \iff \boxed{Ric_{kl} = f\, g_{kl}}$$
$$Ric_{kl} = Ric\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$\underline{\nabla g = 0}$

Then for any $h \in \{1, \dots, n\}$

$$\underline{Ric_{kl;h}} = (f g_{kl})_{\underline{;h}} = f_{;h}\, g_{kl} + f\, \overbrace{g_{kl;h}}^{=0}$$
$$= f_{,h}\, g_{kl}$$
$$Ric_{kl;h} = \frac{\partial Ric_{kl}}{\partial x^h} + \boxed{Ric_{kl} * \Gamma}, \qquad f_{,h} = \frac{\partial f}{\partial x^h}$$

休息到 16:45

$$\boxed{Ric_{kl;h} = f_{,h}\, g_{kl}}$$

$$Ric_{kl;h} = (g^{ij} \underline{R_{ikjl}})_{;h} = \underline{\underline{\overbrace{g^{ij}{}_{;h}}^{=0}}} R_{ikjl} + g^{ij} R_{ikjl;h}$$
$$= g^{ij} \underline{\underline{R_{ikjl;h}}}$$

(Exercise)
$$g^{ij} R_{ikhj;l} = -g^{ij} \underline{R_{ikjh;l}}$$
$$= -(g^{ij} R_{ikjh})_{;l}$$
$$= -Ric_{kh;l}$$

By 2nd Bianchi identity

$$0 = g^{ij} \left(R_{ikjl;h} + R_{iklh;j} + \boxed{R_{ikhj;l}} \right)$$
$$= f_{,h}\, g_{kl} + g^{ij} R_{iklh;j} - f_{,l}\, g_{kh}$$

$$\Rightarrow \quad 0 = f_{,h}\, g_{kl} - f_{,l}\, g_{kh} + g^{ij} R_{iklh;j}, \quad \forall h, k, l = 1, \dots, n$$

Set $k = l$ and sum over k, we have, restricting to p.

$$0 = f_{,h}(p) \sum_{k=1}^{n} g_{kk}(p) - \sum_{k=1}^{n} f_{,k}(p)\, g_{kh}(p) - \sum_{i,j} g^{ij}(p) \sum_{k=1}^{n} R_{kikh;j}(p)$$

$$0 = f_{,h}(p)\sum_{k=1}^{n} g_{kk}(p) - \sum_{k=1}^{n} f_{,k}(p)\, g_{kh}(p) - \sum_{ij} g^{ij}(p)\sum_{k=1}^{n} R_{kikh;j}(p)$$

$$= n f_{,h}(p) - f_{,h}(p) - \sum_{i,k} R_{kikh;i}(p)$$

$$= (n-1) f_{,h}(p) - \sum_{i,k,l}\left(\underline{g^{kl} R_{kilh;i}}\right)(p)$$

$$= (n-1) f_{,h}(p) - \sum_{i} \underline{Ric_{ih;i}}(p)$$

$$= (n-1) f_{,h}(p) - \sum_{i} f_{,i}(p)\, \underline{g_{ih}(p)}$$

$$= (n-1) f_{,h}(p) - f_{,h}(p)$$

$$= \underline{(n-2)} f_{,h}(p)$$

Since $n \geq 3$, we obtain $f_{,h}(p) = 0$, $\forall\, h = 1, \dots, n$.

In conclusion, $\forall p \in M$, $\forall X_p \in T_pM$, we have $X_p(f) = 0$.

$$\Rightarrow f \equiv \text{const}$$

$\square$

Scalar curvature. The scalar curvature S is defined as the trace of the Ricci curvature tensor, i.e.

$$S = tr\, \underline{Ric} \overset{(U,x)}{=} g^{ij} Ric_{ij}$$

$$= tr\left(X \to \underline{\# Ric(X, \cdot)}\right)$$

Recall: $g(\# Ric(X,\cdot), Y) = Ric(X,Y)$

X_p unit tangent vector, $\{X_p, e_2, \dots, e_n\}$ orthonormal basis

$$Ric(X_p, X_p) = tr\, R(\cdot, X_p, \cdot, X_p)$$

$$\langle \cdot, X_p \rangle \quad = \sum_{i=2}^{n} R(e_i, X_p, e_i, X_p)$$

$$= \sum_{i=2}^{n} \langle R(e_i, X_p) X_p, e_i \rangle$$

$$= \sum_{i=2}^{n} \langle R(X_p, e_i) e_i, X_p \rangle, \quad \forall X_p$$

$$\Rightarrow \# Ric(X_p, \cdot) = \sum_{i=2}^{n} R(X_p, e_i) e_i$$

$$X_p \longmapsto \sum_{i=2}^{n} R(X_p, e_i) e_i \quad \text{linear transformation}$$

$$S = tr\, \underline{Ric} = tr\left(X_p \longmapsto \sum_{i=2}^{n} R(X_p, e_i) e_i\right)$$

$S = \mathrm{tr}\,\underline{Ric} = \mathrm{tr}\,(X_p \longmapsto \sum_{i=1} K(X_p, e_i) e_i)$

Rmk $\{e_1, \dots, e_n\}$ orthonormal basis of T_pM, we have

$$S(p) = \mathrm{tr}\,Ric(\cdot,\cdot) = \sum_{i=1}^{n} Ric(e_i, e_i)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} R(e_i, e_j, e_i, e_j)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} K(\Pi_{\{e_i, e_j\}})$$
$$= 2 \underbrace{\sum_{i<j}}_{\binom{n}{2}} K(\Pi_{\{e_i, e_j\}})$$

Rmk. If (M,g) has constant (sectional) curvature k, we have $Ric = \underline{(n-1)k}\, g$, and $S = n(n-1)k$

· If (M,g) is Einstein with constant k, then

$$S = nk$$

Prop. An n ($n \geq 3$) dim'l Rie. mfld (M,g) is Einstein iff $$Ric = \frac{S}{n} g$$

Proof: $\Rightarrow$ by definition

$\Leftarrow$ $Ric = \underset{\substack{\shortparallel \\ f(p)}}{\frac{S}{n}(p)}\, g$ Schur. $\Rightarrow$ $\frac{S}{n} \equiv$ const. $\square$

Rmk. $n=2$ $\quad K(\Pi_p) = \dfrac{Ric(X_p, X_p)}{g(X_p, X_p)} = \dfrac{1}{2} S(p)$

$n=3$ There is no difference in knowing K or Ric.

$\{e_1, e_2, e_3\}$ orthonormal basis of T_pM.

$K(e_i, e_j) = K(\Pi_{\{e_i, e_j\}})$

$$\begin{cases} K(e_1, e_2) + K(e_1, e_3) = Ric(e_1, e_1) \\ K(e_1, e_2) + K(e_2, e_3) = Ric(e_2, e_2) \\ K(e_1, e_3) + K(e_2, e_3) = Ric(e_3, e_3) \end{cases}$$

Reformulated as

$K(e_1,e_3) + K(e_2,e_3) = Ric(e_3,e_3)$

Reformulated as

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} K(e_1,e_2) \\ K(e_2,e_3) \\ K(e_1,e_3)\end{pmatrix} = \begin{pmatrix} Ric(e_1,e_1) \\ Ric(e_2,e_2) \\ Ric(e_3,e_3)\end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2 \neq 0$$

Prop. (M^3, g) is Einstein iff (M^3, g) has constant sectional curvature.

Proof $\square$

Rmk. "$Ric \geq k$", $\forall X$, $Ric(X,X) \geq k\, g(X,X)$

$$\boxed{\frac{Ric(X,X)}{g(X,X)}} \geq k$$

→ Rayleigh quotient.

$$X \mapsto \# Ric(X,\cdot) = \sum_{i=1}^{n} Ric(X,e_i)e_i$$

At $p \in M$, the optimal k is the least eigenvalue of

$$X \mapsto \# Ric(X,\cdot)$$

Lohkamp.

Thm (Ann. Math. 1994) For each manifold M^n, $n \geq 3$,

$\exists$ a complete metric g_M, with

$$-a(n)\, g_M < Ric(g_M) < -b(n)\, g_M$$

with constants $a(n) > b(n) > 0$ depending only on the dim. n

Thm For each manifold M^n, $n \geq 3$, $\exists$ a complete metric g_M with negative Ricci curvature and finite volume

休息到 17:45

Bochner identity. (M,g) Rie. mfld. $f \in C^\infty(M)$

grad f, Hess f $\Delta f = \operatorname{tr} \operatorname{Hess} f$ $\nearrow X_p$

$$\frac{1}{2}\Delta |\operatorname{grad} f|^2 = \underbrace{|\operatorname{Hess} f|^2}_{\geq \frac{1}{n}(\Delta f)^2} + \langle \operatorname{grad}(\Delta f), \operatorname{grad} f\rangle + \underline{Ric}(\operatorname{grad} f, \operatorname{grad} f)$$

Rmk. $|\cdot|$ norm of the corresponding tensors w.r.t g

$\geq \frac{1}{n}(\Delta f)^2$

<u>Rmk</u>. $|\cdot|$ norm of the corresponding tensors w.r.t g

(U,x) $|\text{grad} f|^2 = \langle \text{grad} f, \text{grad} f \rangle = \underline{g_{kl}} g^{ik} \frac{\partial f}{\partial x^i} g^{jl} \frac{\partial f}{\partial x^j}$

$\text{grad} f = \boxed{g^{ij} \frac{\partial f}{\partial x^i}} \frac{\partial}{\partial x^j}$ $\qquad = \underline{g^{jl} \frac{\partial f}{\partial x^l} \frac{\partial f}{\partial x^j}}$

$|\text{Hess} f|^2 = g^{ik} g^{jl} \underline{\text{Hess} f \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)} \text{Hess} f\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$

$= \underline{g^{ik} g^{jl}} \underline{f_{;ij}} f_{;kl}$

(U,x) normal around p. $\qquad \sum_{i,j} \underline{f_{;ij}^2 (p)}$ $\square$

<u>Proof</u>. Prove it pointwisely. $\forall\, p \in M$, pick a normal chart (U,x) of p.

$\frac{1}{2} \Delta \underline{|\text{grad} f|^2} (p) = \frac{1}{2} \underline{g^{kl}} \left(g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}\right)_{;kl} (p)$

$= \frac{1}{2} \sum_k \left(g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}\right)_{;kk} (p)$

$\overset{\nabla g = 0}{=} \frac{1}{2} \sum_k \left(g^{ij} \left(\frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}\right)_{;k}\right)_{;k} (p)$

$\overset{\nabla g = 0}{=} \frac{1}{2} \sum_k g^{ij} \left(\frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}\right)_{;kk} (p)$

$= \frac{1}{2} \sum_{k,i} \left[\left(\frac{\partial f}{\partial x^i}\right)^2\right]_{;kk} (p) = \frac{1}{2} \sum_{k,i} \left(2 \left(\frac{\partial f}{\partial x^i}\right) \left(\frac{\partial f}{\partial x^i}\right)_{;k}\right]_{;k} (p)$

$= \sum_{k,i} \underbrace{\left(\frac{\partial f}{\partial x^i}\right)}_{f_{;i}}{}_{;k} \left(\frac{\partial f}{\partial x^i}\right)_{;k} (p) + \sum_{k,i} \frac{\partial f}{\partial x^i} \left(\frac{\partial f}{\partial x^i}\right)_{;kk} (p)$

$= \sum_{k,i} (f_{;ik})^2 (p) + \sum_{k,i} f_{;i} f_{;ikk} (p)$

$\overset{\text{Ricci ident}}{=} \sum_{k,i} (f_{;ik})^2 (p) + \sum_{k,i} f_{;i} f_{;kik} (p)$

$\overset{\text{Ricci identity}}{=} \sum_{k,i} (f_{;ik})^2 (p) + \sum_{k,i} \underline{f_{;i}} (f_{;kki} + f_{;h} R^h{}_{kik}) (p)$

$= \sum_{k,i} (f_{;ik})^2 (p) + \sum_{k,i} f_{;i} f_{;kki} (p) + \sum_{k,i} f_{;i} f_{;h} R^h{}_{kik} (p)$

$\sum_{k,i} (f_{;ik})^2 (p) = \underline{g^{ij} g^{kl} f_{;ik} f_{;jl}} (p) = |\text{Hess} f|^2 (p)$

$$\sum_{k,i} (f_{;ik})^2(p) = \underline{g^{ij} g^{kl} f_{;ik} f_{;jl}}(p) = |\mathrm{Hess} f|^2(p).$$

$$\sum_{k,i} f_{;i}(p) f_{;kki}(p) = \underline{g^{ij}{}_{(p)} f_{;i}(p) (g^{kl} f_{;kl})_{;j}(p)}$$

$$= \langle \mathrm{grad} f, \mathrm{grad} \Delta f \rangle (p).$$

$$\sum_{k,i} f_{;i} f_{;h} R^{h}{}_{kik}(p) = \sum_{h,k,i} f_{;i} f_{;h} \underbrace{R_{hkik}}(p)$$

$$= \sum_{h,i} f_{;i} f_{;h} (\underline{\textstyle\sum_k R_{hkik}})(p) \qquad \underline{g_{hm}} R^m{}_{kik}(p)$$

$$= \sum_{h,i} f_{;i} f_{;h} \mathrm{Ric}_{hi}(p)$$

$$= \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f)(p).$$

$\square$

$$\underline{|\mathrm{Hess} f|^2 \geq \frac{1}{n}(\Delta f)^2}$$

(U, x) around p. $\sum_{i,j} (f_{;ij})^2(p) \qquad \frac{1}{n}\left(\sum_i f_{;ii}\right)^2(p)$

$$\geq \sum_i (f_{;ii})^2(p) \geq \frac{1}{n}\left(\sum_i f_{;ii}\right)^2(p).$$

Cauchy-Schwarz.

Bochner identity $\Rightarrow$

$$\frac{1}{2}\Delta|\mathrm{grad} f|^2 \geq \frac{1}{n}(\Delta f)^2 + \underline{\langle \mathrm{grad} f, \mathrm{grad}\Delta f \rangle} + \mathrm{Ric}(\mathrm{grad} f)$$

<u>Thm</u> (M^n, g), $p \in M$. $\mathrm{Ric}(X,X)(p) \geq K(p)\, g(X,X)(p)$, $\forall X \in \Gamma(TM)$

$$\Leftrightarrow \frac{1}{2}\Delta|\mathrm{grad} f|^2_{(p)} - \langle \mathrm{grad} f, \mathrm{grad}\Delta f \rangle_{(p)} \geq \underline{\frac{1}{n}(\Delta f)^2_{(p)}} + K(p)|\mathrm{grad} f|^2_{(p)}$$

$\forall f \in C^\infty(M)$

$\Rightarrow \checkmark$

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