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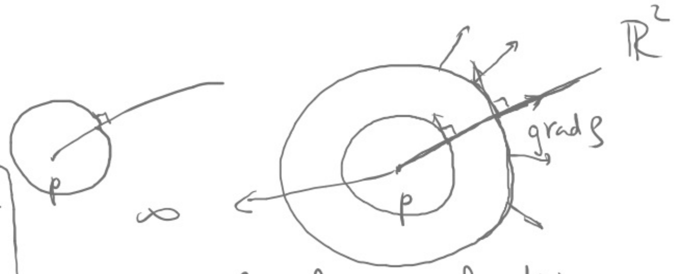
Riemannian Geometry Lecture 29

Splitting theorem. Let (M^n, g) be a complete noncompact Rie mfld with $Ric \geq 0$. Suppose (M^n, g) contains a line. Then (M^n, g) isometric to $\underset{\sim}{M'} \times \underline{\mathbb{R}}$, $\underline{M'}$ is complete Rie mfld with $Ric \geq 0$

Gauss' lemma.

$$dr^2 + \boxed{\ f(r,\theta)\ } d\theta^2$$

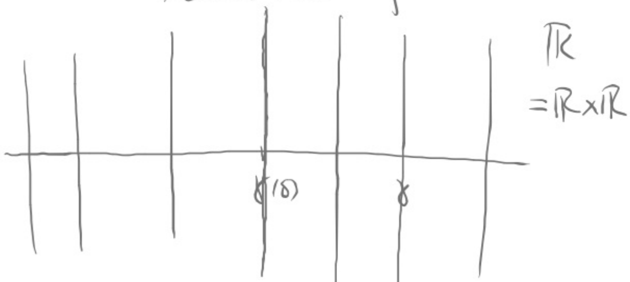
"splitting"



level set of distance

$\rho(\cdot) = d(p, \cdot)$

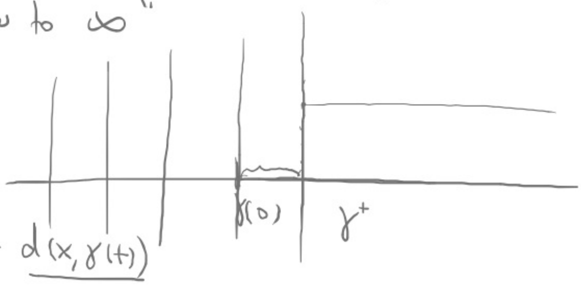
$|\mathrm{grad}\, \rho| = 1$



Busemann function. "distance to ∞"

$b^{\gamma^+}: M \to \mathbb{R}$

$x \mapsto b^{\gamma^+}(x) = \lim_{t\to\infty} \left(t - \underline{d(x, \gamma(t))}\right)$



$\Rightarrow b^{\gamma^+}$ 1-Lipschitz.

Laplacian comparison $\Rightarrow \Delta b^{\gamma^+} \geq 0$ in the sense of barrier.

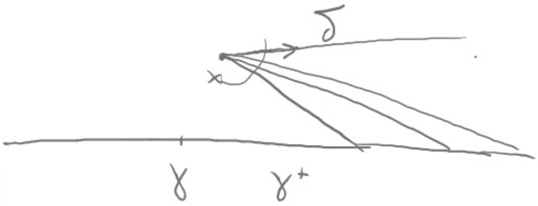
$\Rightarrow b^{\gamma^+}$ is smooth & $\underline{\Delta b^{\gamma^+} = 0}$, $b^{\gamma^+} + b^{\gamma^-} = 0$

<u>Prop</u>: $|\mathrm{grad}\, b^{\gamma^+}| = 1$

<u>Proof</u>: b^{γ^+} is 1-Lipschitz $\Rightarrow$

$\underline{|b^{\gamma^+}(x) - b^{\gamma^+}(y)| \leq d(x,y)}$

$\Rightarrow |\mathrm{grad}\, b^{\gamma^+}| \leq 1$



$\Rightarrow |\mathrm{grad}\, b^{\gamma^+}| \leq 1$

At x, $v = \dot{\gamma}(0)$

$$|\mathrm{grad}\, b^{\gamma^+}| = |\mathrm{grad}\, b^{\gamma^+}| \cdot |v| \geq \langle \mathrm{grad}\, b^{\gamma^+}, v \rangle$$

$$= v(b^{\gamma^+}) = \lim_{t \to 0} \frac{b^{\gamma^+}(\gamma(t)) - b^{\gamma^+}(\overset{p}{\gamma(0)})}{t} = 1. \quad \square$$

$$b^{\gamma^+}(\gamma(t)) = b^{\gamma^+}(x) + t$$

Consequence ① $|\mathrm{grad}\, b^{\gamma^+}| \equiv 1$, $b^{\gamma^+} : M \longrightarrow \mathbb{R}$

$\forall\, a \in \mathbb{R}$. $\quad (b^{\gamma^+})^{-1}(a) := \{ x \in M \mid b^{\gamma^+}(x) = a \} = V_a$

level set

Implicit function theorem $\Rightarrow$ V_a is a (n-1)-dim'l regular submanifold.

$$i : V_a \longrightarrow M$$

Consequence ② $\underbrace{\Delta b^{\gamma^+} = 0 + |\mathrm{grad}\, b^{\gamma^+}| \equiv 1}_{Ric \geq 0} \Rightarrow \mathrm{grad}\, b^{\gamma^+}$ is parallel.

Bochner formula: $f \in C^{\infty}(M)$

$$\frac{1}{2} \Delta \underbrace{|\mathrm{grad} f|^2}_{\equiv 1} = |\mathrm{Hess} f|^2 + \langle \mathrm{grad} f, \mathrm{grad}\, \Delta f \rangle + \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f)$$

Set $f = b^{\gamma^+} \Rightarrow 0 \geq |\mathrm{Hess}\, b^{\gamma^+}|^2 \Rightarrow |\mathrm{Hess}\, b^{\gamma^+}| = 0$.

For any $X, Y \in \Gamma(TM)$, we have $\mathrm{Hess}\, b^{\gamma^+}(X, Y) = 0$

$$0 = \mathrm{Hess}\, b^{\gamma^+}(X, Y) = \nabla^2 b^{\gamma^+}(X, Y) = \nabla_Y (\nabla b^{\gamma^+})(X)$$

$$= \nabla_Y (\nabla_X b^{\gamma^+}) - \nabla b^{\gamma^+} (\nabla_Y X)$$

$$= Y(X b^{\gamma^+}) - \nabla_Y X (b^{\gamma^+})$$

$$= Y \langle X, \mathrm{grad}\, b^{\gamma^+} \rangle - \langle \nabla_Y X, \mathrm{grad}\, b^{\gamma^+} \rangle$$

$$= \langle X, \nabla_Y \mathrm{grad}\, b^{\gamma^+} \rangle, \quad \forall\, X, Y$$

$\Rightarrow \nabla \mathrm{grad}\, b^{\gamma^+} = 0 \Rightarrow \mathrm{grad}\, b^{\gamma^+}$ is parallel $\square$

Cor. V_a is totally geodesic submanifold.

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Def. $(M^n, g) \subset (\overline{M}^{n+1}, \overline{g})$ be a submanifold is totally geodesic if $\forall\, p \in M$, $\forall\, v \in T_pM$, the geodesic γ eminating from p with $\gamma'(0) = v$ lie in M

g induced by $\overline{g}$

Remark. If γ is a geodesic in $(\overline{M}, \overline{g})$ ~~then~~ and $\gamma \subset M$ then γ is also a geodesic in M.

$\overline{g} \to \overline{\nabla}$ Levi-Civita connection $\overline{M}$

$g \mapsto \nabla$ M

$\forall\, X, Y \in \Gamma(TM)$, $\nabla_X Y := \boxed{\pi(\overline{\nabla}_X Y)}$ Excerise

$p \in M$. $T_p\overline{M} = T_pM \oplus N_pM$

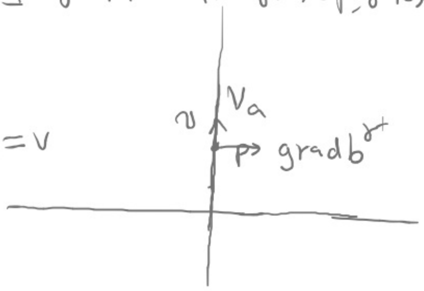
$$\overline{\nabla}_{\dot\gamma}\dot\gamma = 0 \Rightarrow \nabla_{\dot\gamma}\dot\gamma = 0$$

Cor V_a is a totally geodesic.

Proof. $p \in V_a$ $\forall\, v \in T_pV_a$ $\exists\, \gamma$ in M. $\gamma(0) = p$, $\gamma'(0) = v$

(V_a, ∇) $\exists\, \tilde\gamma \subset M$ s.t. $\tilde\gamma(0) = p$, $\tilde\gamma'(0) = v$

$(M, \overline{\nabla})$ $\nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = 0 \overset{?}{\Rightarrow} \overline{\nabla}_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = 0$



$$\underbrace{\overline{\nabla}_{\dot{\tilde\gamma}}\dot{\tilde\gamma} - \nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma}}_{0} = \overline{\nabla}_{\dot{\tilde\gamma}}\dot{\tilde\gamma} - \pi(\overline{\nabla}_{\dot{\tilde\gamma}}\dot{\tilde\gamma})$$

$$= \underbrace{\langle \overline{\nabla}_{\dot{\tilde\gamma}}\dot{\tilde\gamma}, \operatorname{grad} b^{\gamma+}\rangle}\, \operatorname{grad} b^{\gamma+}$$

$\forall\, X, Y \in \Gamma(TV_a)$

$$\langle \overline{\nabla}_X Y, \underline{\operatorname{grad}} b^{\gamma+}\rangle = X\overbrace{\langle Y, \operatorname{grad} b^{\gamma+}\rangle}^{\equiv 0} - \langle Y, \underbrace{\nabla_X \operatorname{grad} b^{\gamma+}}_{=0}\rangle = 0.$$

$$\Rightarrow \overline{\nabla}_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = \nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = 0$$ $\square$

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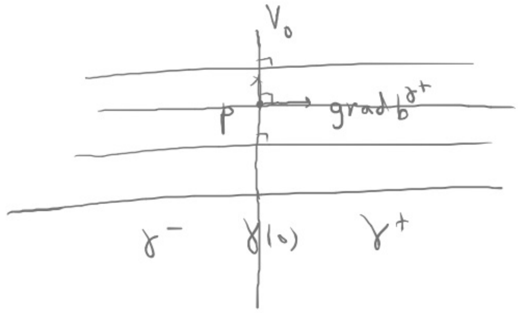
$b^{\gamma^+}(x)$

<u>Aim</u> $(M^n, g) \cong \underline{V_0 \times \mathbb{R}}$



$b^{\gamma^+}(\gamma(0)) = 0 \Rightarrow \gamma(0) \in V_0$

$\phi : V_0 \times \mathbb{R} \longrightarrow (M^n, g)$

$(p, t) \longmapsto \phi(p, t) = \exp_p\left(t \ \mathrm{grad}\, b^{\gamma^+}(p)\right)$

① ϕ diffeomorphism ? ② isometric $\phi^* g$

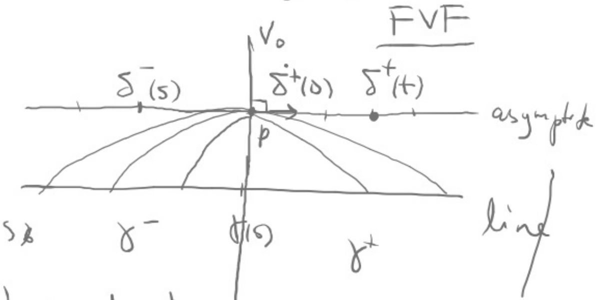
① ϕ is locally diffeomorphism.

ϕ surjective, injective

Given p, $t \mapsto \exp_p\left(t \ \mathrm{grad}\, b^{\gamma^+}(p)\right)$ is a geodesic line

FVF

two rays. δ^+, δ^-



$t + s \overset{?}{=} d(\delta^+(t), \delta^-(s))$, $\forall t, s$

$t + s = d(p, \delta^+(t)) + d(p, \delta^-(s)) \geq \underline{d(\delta^+(t), \delta^-(s))}$

b^{γ^+} is 1-Lipschitz.

$\geq b^{\gamma^+}(\delta^+(t)) - b^{\gamma^+}(\delta^-(s))$

$b^{\gamma^+} + b^{\gamma^-} = 0$

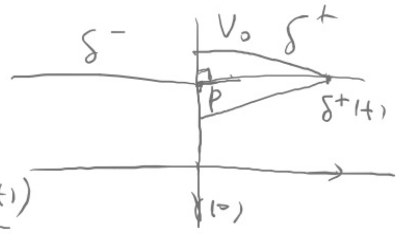
$= b^{\gamma^+}(\delta^+(t)) + b^{\gamma^-}(\delta^-(s))$

$= \underset{\overset{\shortparallel}{0}}{b^{\gamma^+}(p)} + t + \underset{\overset{\shortparallel}{-b^{\gamma^+}(p) = 0}}{b^{\gamma^-}(p)} + s$

$= t + s$

$\Rightarrow t + s = d(\delta^+(t), \delta^-(s))$

$\Rightarrow$ line geodesic



$\forall y \in V_0$, $\delta^+(t)$

$d(y, \delta^+(t)) \overset{?}{\geq} t = \underline{d(\delta^+(0), \delta^+(t))}$

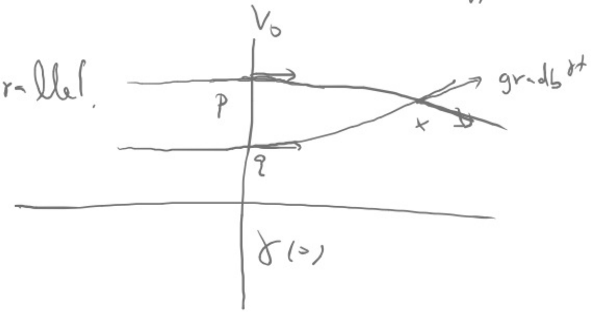
$d(y, \delta^+(t)) \geq b^{\gamma^+}(\delta^+(t)) - b^{\gamma^+}(y)$

$$d(y, \gamma^+(t)) \overset{\checkmark}{\geq} b^{\gamma^+}(\gamma^+(t)) - b^{\gamma^+}(y) = t$$

$\Longrightarrow t \mapsto \exp_p t \operatorname{grad} b^{\gamma^+}(p)$ is a line. $\Rightarrow \phi$ is a locall diffeom.

⊗ injective

$\phi(p, t_1) = \phi(q, t_2)$ $\operatorname{grad} b^{\gamma^+}$ is parallel.

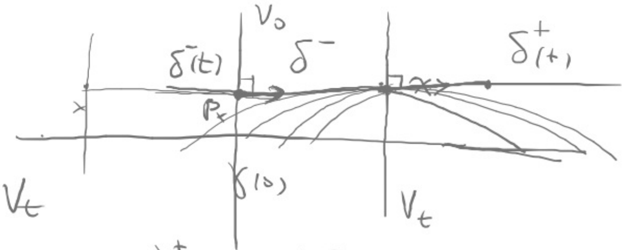


⊗ <u>surjective</u>

$\forall x \in M$, $b^{\gamma^+}(x) = t$

<u>Claim.</u> $\exists$ a line pass through x, which is vertical to V_t

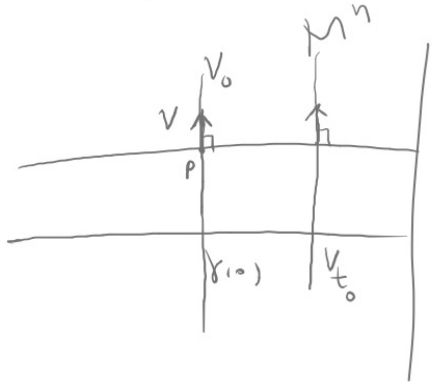
<u>Exercise</u>



$b^{\gamma^+}(x) \begin{cases} >0 \\ <0 \end{cases}$

$\Longrightarrow \phi$ is a diffeomorphism. □

$$\phi : V_0 \times \mathbb{R} \longrightarrow (M^n, g)$$



$$\underline{\phi^* g = g_{V_0} \oplus g_{\text{eucl}}}$$

$$\forall \ W \in T_{(p,t)}(V_0 \times \mathbb{R})$$

$$W = W_1 \oplus W_2$$

Given $p \in V_0$, $t \mapsto (p, t)$.

$$\phi(p, t) = \underline{\exp_p(t \operatorname{grad} b^{\gamma^+})}$$

$\phi^* g(W_2, W_2) = g_{V_0} \oplus g_{\text{eucl}}(W_2, W_2)$

Remains to show

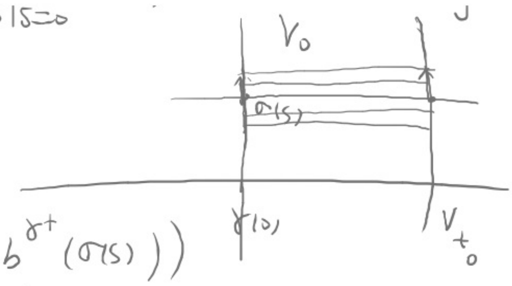
$s \longmapsto (\sigma(s), t_0)$ $\quad \left| \frac{d}{ds}\Big|_{s=0} \phi(\sigma(s), t_0) \right|_g = |\sigma'(s)|_{g_{V_0}}$

$\sigma(0) = p \quad \sigma'(0) = v$

$$\| \qquad \left| \frac{d}{ds}\Big|_{s=0} \exp_{\sigma(s)}(t_0 \operatorname{grad} b^{\gamma^+}(\sigma(s))) \right|_g$$

$b^{\gamma^+}\left(\underline{\exp_{\sigma(s)}(t_0 \operatorname{grad} b^{\gamma^+}(\sigma(s)))}\right)$

$b^{\gamma^+}\left(\underline{\exp_{\sigma(s)}\left(t_0 \operatorname{grad} b^{\gamma^+}(\sigma(s))\right)}\right) \Big| \quad 0 \le |s| \ge 0$



$= b^{\gamma^+}(\gamma^+(t_0)) = t_0$

$$F(t,s) = \exp_{\sigma(s)}\left(t \operatorname{grad} b^{\gamma^+}(\sigma(s))\right)$$

$$U(t) = \frac{\partial}{\partial s}\Big|_{s=0} F(t,s) \qquad t \in (-\infty, \infty) \quad s \in (-\varepsilon, \varepsilon)$$

$$U(0) = \frac{\partial}{\partial s}\Big|_{s=0} \sigma(s) = v. \quad U(t_0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\sigma(s)}\left(t_0 \operatorname{grad} b^{\gamma^+}(\sigma(s))\right)$$

Remains to show $|U(t_0)| = |U(0)|$

$$\nabla_T \langle U(t), U(t)\rangle = 2\langle U(t), \nabla_T U(t)\rangle = 0$$

$$\nabla_T U(t) = \nabla_T \frac{\partial}{\partial s}\Big|_{s=0} F(t,s) = \tilde{\nabla}_{\frac{\partial}{\partial s}}\Big|_{s=0} \underline{\frac{\partial}{\partial t} F(t,s)}$$

$$= \tilde{\nabla}_{\frac{\partial}{\partial s}}\Big|_{s=0} \operatorname{grad} b^{\gamma^+}(\sigma(s)) = 0$$

$\Rightarrow \quad |U(t)| \equiv \text{const.} \quad \Rightarrow \quad |U(0)| = |U(t_0)|.$

$\Rightarrow \quad \phi$ is an isometry

$$(M^n, g) \cong \underline{V_0} \times \underline{\mathbb{R}}$$

V_0 cpl $\quad Ric \ge 0$? $\qquad$ totally geodesic

$$\nabla_X Y = \overline{\nabla}_X Y$$

下课. $\qquad \square$